Creelman

Exam Questions

Mathematics Specialist

ATAR COURSE UNITS 3 & 4

Exam questions, arranged in topics, with model answers

2022 EDITION

TIM OATES

Unit 3 Content

An understanding of the Year 11 content is assumed knowledge for students in Year 12. It is recommended that students studying Unit 3 and Unit 4 have completed Unit 1 and Unit 2. The examinable content for Unit 3 is as follows:

Topic 3.1: Complex numbers (18 hours)

Cartesian forms

3.1.1 review real and imaginary parts $\text{Re}(z)$ and $\text{Im}(z)$ of a complex number z
3.1.2 review Cartesian form
3.1.3 review complex arithmetic using Cartesian forms

Complex arithmetic using polar form

3.1.4 use the modulus $|z|$ of a complex number z and the argument $\text{Arg}(z)$ of a non-zero complex number z and prove basic identities involving modulus and argument
3.1.5 convert between Cartesian and polar form
3.1.6 define and use multiplication, division, and powers of complex numbers in polar form and geometric interpretation of these
3.1.7 prove and use De Moivre's theorem for integral powers

The complex plane (The Argand plane)

3.1.8 examine and use addition of complex numbers as vector addition in the complex plane
3.1.9 examine and use multiplication as a linear transformation in the complex plane
3.1.10 identify subsets of the complex plane determined by relations such as

$$|z-3i| \leq 4, \; \frac{\pi}{4} \leq Arg(z) \leq \frac{3\pi}{4} \text{ and } |z-1| = 2|z-i|$$

Roots of complex numbers

3.1.11 determine and examine the nth roots of unity and their location on the unit circle
3.1.12 determine and examine the nth roots of complex numbers and their location in the complex plane

Factorisation of polynomials

3.1.13 prove and apply the factor theorem and the remainder theorem for polynomials
3.1.14 consider conjugate roots for polynomials with real coefficients
3.1.15 solve simple polynomial equations

Topic 3.2: Functions and sketching graphs (16 hours)

Functions

3.2.1 determine when the composition of two functions is defined
3.2.2 determine the composition of two functions
3.2.3 determine if a function is one-to-one
3.2.4 find the inverse function of a one-to-one function
3.2.5 examine the reflection property of the graphs of a function and its inverse

Sketching graphs

3.2.6 use and apply $|x|$ for the absolute value of the real number x and the graph of $y = |x|$
3.2.7 examine the relationship between the graph of

$y = f(x)$ and the graphs of $y = \frac{1}{f(x)}$, $y = |f(x)|$ and $y = f(|x|)$

3.2.8 sketch the graphs of simple rational functions where the numerator and denominator are polynomials of low degree

Topic 3.3: Vectors in three dimensions (21 hours)

The algebra of vectors in three dimensions

3.3.1 review the concepts of vectors from Unit 1 and extend to three dimensions, including introducing the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$
3.3.2 prove geometric results in the plane and construct simple proofs in 3 dimensions

Vector and Cartesian equations

3.3.3 introduce Cartesian coordinates for three dimensional space, including plotting points and equations of spheres
3.3.4 use vector equations of curves in two or three dimensions involving a parameter and determine a 'corresponding' Cartesian equation in the two-dimensional case
3.3.5 determine a vector equation of a straight line and straight line segment, given the position of two points or equivalent information, in both two and three dimensions
3.3.6 examine the position of two particles, each described as a vector function of time, and determine if their paths cross or if the particles meet
3.3.7 use the cross product to determine a vector normal to a given plane
3.3.8 determine vector and Cartesian equations of a plane

Systems of linear equations

3.3.9 recognise the general form of a system of linear equations in several variables, and use elementary techniques of elimination to solve a system of linear equations
3.3.10 examine the three cases for solutions of systems of equations – a unique solution, no solution, and infinitely many solutions – and the geometric interpretation of a solution of a system of equations with three variables

Vector calculus

3.3.11 consider position vectors as a function of time
3.3.12 derive the Cartesian equation of a path given as a vector equation in two dimensions, including ellipses and hyperbolas
3.3.13 differentiate and integrate a vector function with respect to time
3.3.14 determine equations of motion of a particle travelling in a straight line with both constant and variable acceleration
3.3.15 apply vector calculus to motion in a plane, including projectile and circular motion

Unit 4 Content

An understanding of the Year 11 content is assumed knowledge for students in Year 12. It is recommended that students studying Unit 3 and Unit 4 have completed Unit 1 and Unit 2. The examinable content for Unit 4 is as follows:

Topic 4.1: Integration and applications of integration (20 hours)

Integration techniques

4.1.1 integrate using the trigonometric identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x),\ \cos^2 x = \frac{1}{2}(1 + \cos 2x) \text{ and } 1 + \tan^2 x = \sec^2 x$$

4.1.2 use substitution $u = g(x)$ to integrate expressions of the form $f(g(x))g'(x)$

4.1.3 establish and use the formula $\int \frac{1}{x}\,dx = \ln|x| + c$ for $x \neq 0$

4.1.4 use partial fractions where necessary for integration in simple cases

Applications of integral calculus

4.1.5 calculate areas between curves determined by functions

4.1.6 determine volumes of solids of revolution about either axis

4.1.7 use technology with numerical integration

Topic 4.2: Rates of change and differential equations (20 hours)

Applications of differentiation

4.2.1 use implicit differentiation to determine the gradient of curves whose equations are given in implicit form

4.2.2 examine related rates as instances of the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

4.2.3 apply the incremental formula $\partial y \approx \frac{dy}{dx}\,\partial x$ to differential equations

4.2.4 solve simple first order differential equations of the form $\frac{dy}{dx} = f(x)$; differential equations of the form $\frac{dy}{dx} = g(y)$; and, in general, differential equations of the form $\frac{dy}{dx} = f(x)\,g(y)$, using separation of variables

4.2.5 examine slope (direction or gradient) fields of a first order differential equation

4.2.6 formulate differential equations, including the logistic equation that will arise in, for example, chemistry, biology and economics, in situations where rates are involved

Modelling motion

4.2.7 consider and solve problems involving motion in a straight line with both constant and non-constant acceleration, including simple harmonic motion and the use of expressions, $\frac{dv}{dt}$, $v\frac{dv}{dx}$ and $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ for acceleration

Topic 4.3: Statistical inference (15 hours)

Sample means

4.3.1 examine the concept of the sample mean $\overline{X}$ as a random variable whose value varies between samples where X is a random variable with mean μ and the standard deviation σ

4.3.2 simulate repeated random sampling, from a variety of distributions and a range of sample sizes, to illustrate properties of the distribution of $\overline{X}$ across samples of a fixed size n , including its mean μ its standard deviation $\frac{\sigma}{\sqrt{n}}$ (where μ and σ are the mean and standard deviation of X), and its approximate normality if n is large

4.3.3 simulate repeated random sampling, from a variety of distributions and a range of sample sizes, to illustrate the approximate standard normality of $\frac{\overline{X}-\mu}{\frac{s}{\sqrt{n}}}$ for large samples ($n \geq 30$), where s is the sample standard deviation

Confidence intervals for means

4.3.4 examine the concept of an interval estimate for a parameter associated with a random variable

4.3.5 examine the approximate confidence interval $\left(\overline{X}-\frac{zs}{\sqrt{n}}, \overline{X}+\frac{zs}{\sqrt{n}}\right)$ as an interval estimate for the population mean μ, where z is the appropriate quantile for the standard normal distribution

4.3.6 use simulation to illustrate variations in confidence intervals between samples and to show that most but not all confidence intervals contain μ

4.3.7 use $\bar{x}$ and s to estimate μ and σ to obtain approximate intervals covering desired proportions of values of a normal random variable, and compare with an approximate confidence interval for μ

The following table shows the number of questions set on each topic in both the calculator free (CF) and calculator assumed (CA) sections of the paper over the last six years. The table indicates the overall emphasis examiners have placed on individual topics in the ATAR examinations. Note however, there is no guarantee that this pattern will be be continued in subsequent examinations. Several topics have additional questions included from other sources to give the student more complete revision assistance.

TOPIC		2016	2016 S	2017	2018	2019	2020	2021	TOTAL
Cartesian and Polar Forms	CF	2		1	3			1	4
	CA			10		12	11, 15	11a,b,c	5
Powers and Roots of Complex Numbers	CF					9	8	7	3
	CA	14		19	10		13		4
Graphing in the Complex Plane	CF			6					1
	CA	10			11	10	10	11d	5
Solving Polynomial Equations	CF	3		2	7	2		6	5
	CA								0
Composite and Inverse Functions	CF	1, 8c,d		4	1	4, 5		2	7
	CA								0
Absolute Value and Rational Functions	CF	8a,b		5	4	7	3, 5	4	7
	CA	12		16a	14		21		4
Vectors	CF	7		7	8		2		4
	CA				17	16, 19	20a	16	5
Applications of Vectors in Three Dimensions	CF								0
	CA	20		14					2
Vector Calculus in Two Dimensions	CF								0
	CA	16		15	19	13	12	19	6
Systems of Equations	CF	6		2			4		3
	CA		8					14	2
Implicit Differentiation	CF						6a,b		1
	CA	17a		12a	20			18	4
Related Rates	CF								0
	CA	15	11a	18			20b	9	5
Integration Techniques	CF	4			6	3	6c,d, 7	3, 5	7
	CA	9		16b					2
Integration Techniques and Trigonometric Functions	CF	5		3	5, 9	1, 6	1		7
	CA								0
Area and Volume	CF			8		8		8	3
	CA	13, 17b,c		12b	15		9, 16	13	7
Differential Equations	CF								0
	CA	18		11, 17c,d	18	11, 17	19	10	8
Modelling Motion	CF								0
	CA	11		17a,b	13	18	14	12	6
Sample Means and Probability Distributions	CF								0
	CA	19a,b,c,d		9	12	14	18a,b	15a,b	6
Sample Means and Confidence Intervals	CF								0
	CA	19e		13	16	15	17, 18c	15c, 17	8

Cartesian and Polar Forms

Chapter

1

1. **[7 marks]**

(MSPEC 2016:CF02)

Give exact expressions for each of the following in the form $a + bi$:

(a) $\dfrac{\overline{2+i}}{(1-i)^2}$. [3]

(b) $(\sqrt{3}-i)^5$. [4]

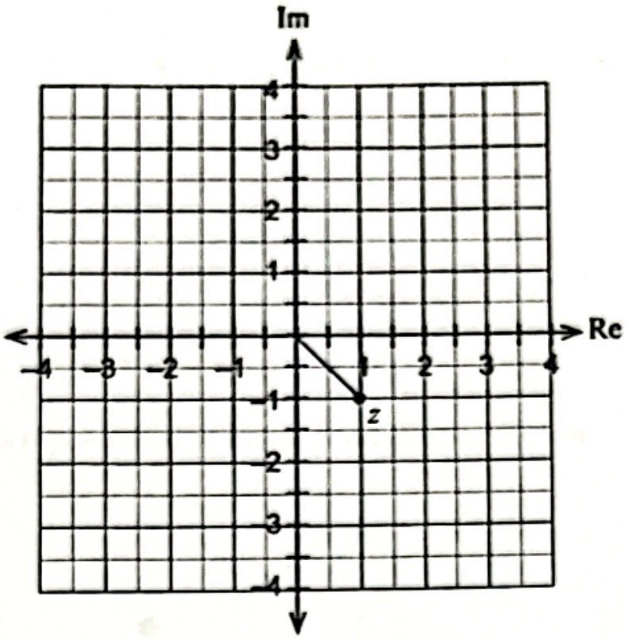
2. **[4 marks]** (MSPEC 2017:CF1)

Let $z = a - bi$, where $a > 0$, $b > 0$. Consider $w = z + i\bar{z}$.

Determine the possible value(s) for $\arg(w)$.

3. **[7 marks]** *(MSPEC 2017:CA10)*

Consider $z = 1 - i$ shown in the complex plane below.



(a) Express z in polar form. [1]

(b) Hence express z^2, z^3 and z^4 in exact polar form. [2]

(c) Sketch the complex numbers z^2, z^3, z^4 as vectors in the given Argand diagram. [2]

Consider the geometric transformation(s) applied to transform $z \rightarrow z^2 \rightarrow z^3 \rightarrow z^4$ etc.

(d) Describe the geometric transformation(s) performed by successive multiplication by z. [2]

4. **[9 marks]** (MSPEC 2018:CF3)

(a) Let $z = a + bi$ be any complex number.

[3]

Obtain an equation relating a, b given that $\text{Re}\left(\frac{z-i}{z}\right) = 0$.

(b) Let $z = r\,\text{cis}\theta$ be any complex number. Obtain an expression for:

(i) $\frac{2i}{\bar{z}}$ in terms of r, θ. [3]

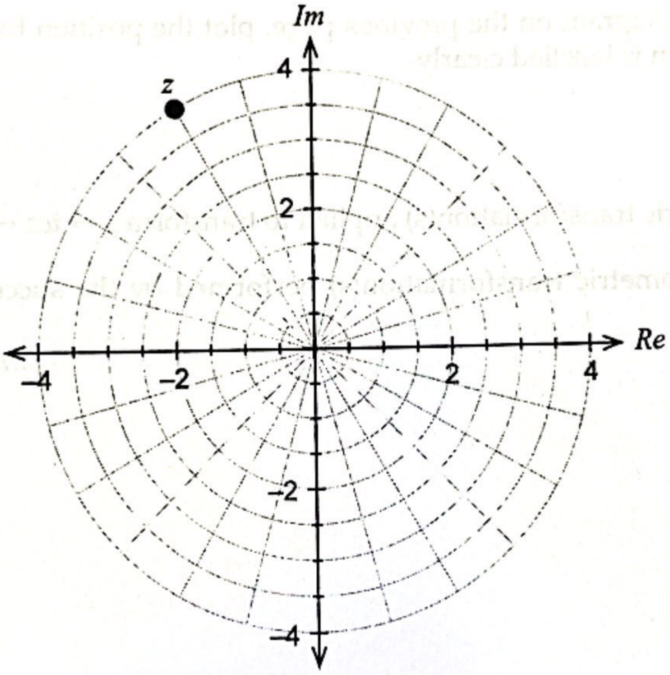
(ii) $\arg(z + r)$ in terms of θ. [3]

5. **[10 marks]** *(MSPEC 2019:CA12)*

Let $w = \dfrac{1-i}{2\sqrt{2}}$.

(a) Express w in the form $w = r\,cis\theta$, where $-\pi < \theta \leq \pi$. [2]

The complex number z is represented in the Argand diagram below.



(b) Express z exactly in the form $z = a + bi$. [2]

7. **[6 marks]** (MSPEC 2020:CA15)

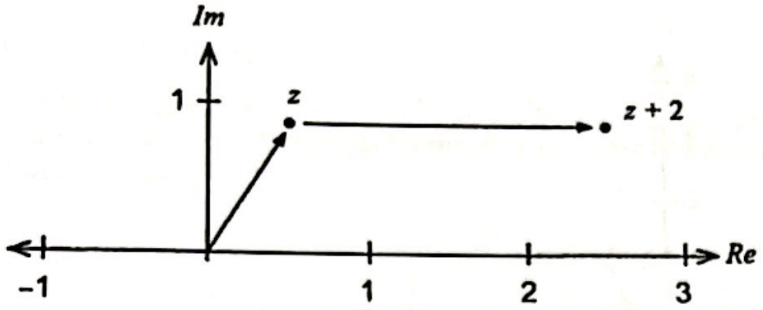
Let $z = r\,cis\theta$ be a complex number such that $\frac{\pi}{2} < \theta < \pi$.

(a) Express in terms of r and θ the complex number $\frac{\bar{z}}{-\sqrt{2}(i+1)}$. [3]

(b) Express $a = Arg(z - ri)$ in terms of θ where $0 < a < 2\pi$. [3]

8. **[4 marks]** (MSPEC 2021:CF01)

The Argand diagram below shows the complex numbers z and $z + 2$ where $z = cis\left(\frac{\pi}{3}\right)$.



Determine the exact value for:

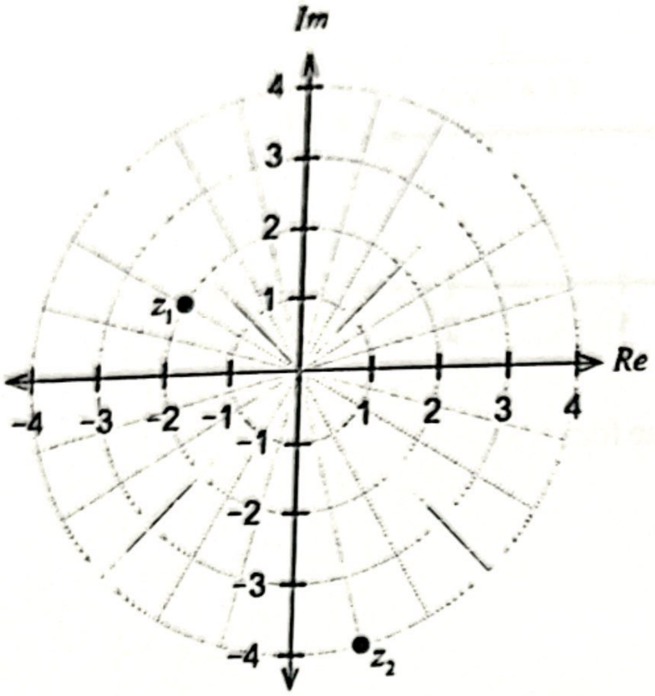
(a) $Arg(-z)$. [1]

(b) $|z + 2|$. [3]

(MSPEC 2021:CA11a,b,c)

9. **[5 marks]**

Two complex numbers z_1 and z_2 are shown in the Argand plane below.



(a) Write the expression for z_1 in exact polar form. [2]

(b) Write the expression for z_1 in exact Cartesian form. [1]

(c) Plot the complex number iz_2 on the Argand diagram above. [2]

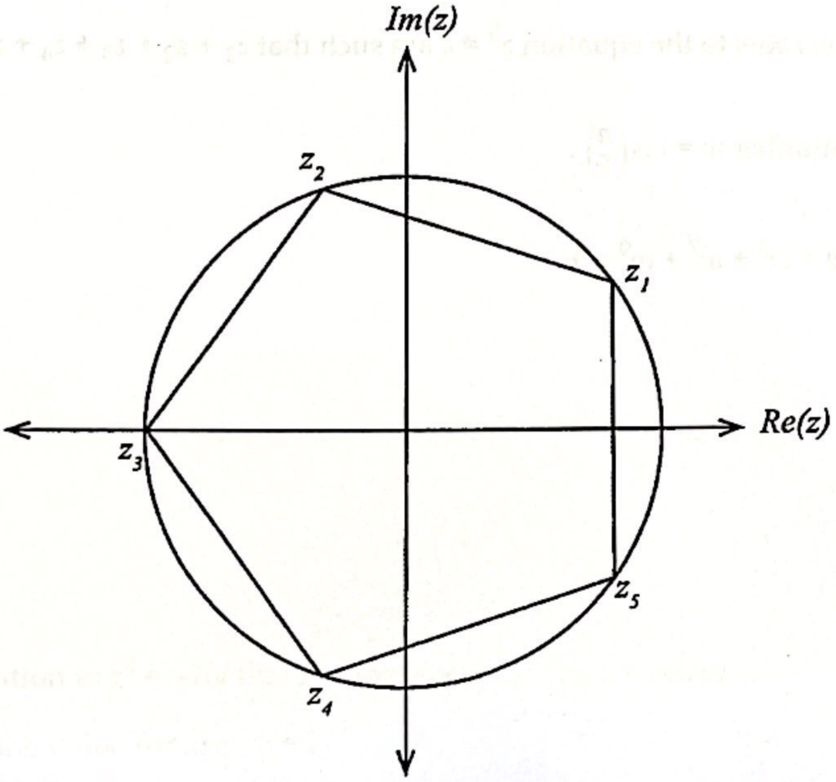
Powers and Roots of Complex Numbers

Chapter

2

1. **[9 marks]** *(3CDMAS 2015:CF8)*

The points given by the complex numbers z_1, z_2, z_3, z_4 and z_5 form a regular pentagon in the complex plane such that each complex number is a solution to the equation $z^5 = k$. It is known that $z_5 = 2 \text{ cis } (-36°)$.



(a) Determine the value of the constant k. [2]

(b) If $z_2 = 2\text{cis}(x)$, determine the exact value for the real number x, where x is in radians. [3]

1. (cont)

(c) Show that $z_4 = \bar{z}_2$. [1]

It is found that the roots to the equation $z^5 = k$ are such that $z_1 + z_2 + z_3 + z_4 + z_5 = 0$.

Let the complex number $w = \text{cis}\left(\frac{\pi}{5}\right)$.

(d) Prove that $w + w^3 + w^7 + w^9 = 1$. [3]

2. **[7 marks]** *(MSPEC 2016:CA14)*

Consider the complex equation $z^4 = -16i$.

(a) Solve the equation giving all solutions in the form $r \operatorname{cis}\theta$ where $-\pi < \theta \leq \pi$. [4]

Let w be the solution to $z^4 = -16i$ that has the least positive argument.

(b) Determine the value for $\arg(w + 2)$. [3]

(MSPEC 2017:CA19)

3. **[7 marks]**

Consider the complex equation $2z^6 = 1 + \sqrt{3}\,i$.

(a) Solve the above equation, giving solutions in polar form $r\text{cis}\theta$ where $0 < \theta < \frac{\pi}{2}$. [4]

Now consider the equation $2z^n = 1 + \sqrt{3}\,i$, where n is a positive integer.

(b) If $2z^n = 1 + \sqrt{3}\,i$ has roots so that there are exactly 3 roots (and only 3) that lie within the first quadrant of the complex plane, determine the possible value(s) of n. Justify your answer.

[3]

4. **[4 marks]** *(MSPEC 2018:CA10)*

Consider the complex number $z = \sqrt{3} + i$.

Show that, for all positive integers n, $(z)^n - (\bar{z})^n = 2^{n+1} \sin\left(\frac{n\pi}{6}\right) i$.

(MSPEC 2019:CF9)

5. **[4 marks]**

Consider the complex equation $z^n - 1 = 0$, where n is any positive integer $n \geq 3$.

If the roots are designated as $z_0, z_1, z_2, \ldots, z_{n-1}$, then determine the exact value for the product of the roots $p = z_0 \times z_1 \times z_2 \times \ \ldots \times z_{n-1}$.

6. **[3 marks]** *(MSPEC 2020:CF8)*

Consider the complex sum: $\sum_{n=1}^{2020} ni^n = 1i^1 + 2i^2 + 3i^3 + \ldots + 2020i^{2020}$

Express the value of this sum in the form $r\ cis\theta$ where $-\pi < \theta \leq \pi$.

7. **[4 marks]** *(MSPEC 2020:CA13)*

Solve the equation $z^4 = 8\sqrt{3} + 8i$ giving exact solutions in the form $r\ cis\theta$ where $-\pi < \theta \leq \pi$.

8. **[5 marks]** (MSPEC 2021:CF07)

The number 2021 can be expressed as a product of two consecutive prime numbers: $43 \times 47 = 2021$.

Consider the complex equation $z^{43} = 1$.

(a) Write an expression for the roots of $z^{43} = 1$. [2]

Let w be any one of the roots of the equation $z^{43} = 1$.

(b) How many of these roots will also be a solution of the equation $z^{47} = 1$? Justify your answer. [3]

Chapter

3

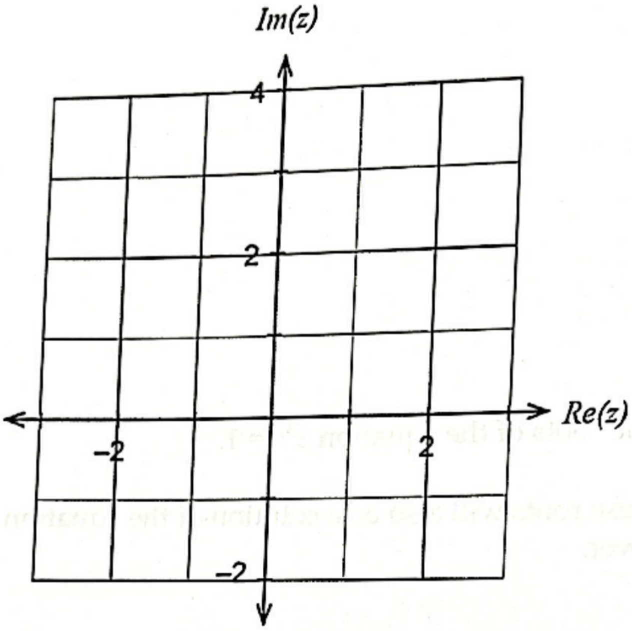
Graphing in the Complex Plane

(MSPEC 2016:CA10)

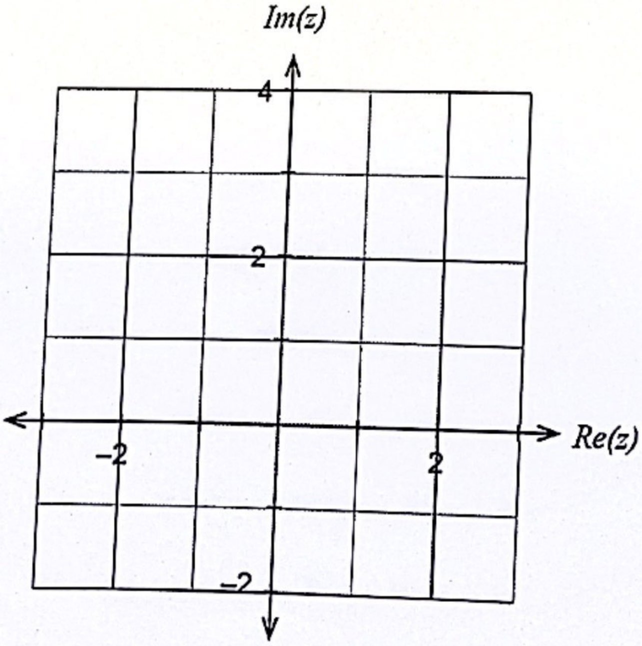
1. **[9 marks]**

On the Argand planes below sketch the locus of the complex number $z = x + iy$ given by:

(a) $|z - 2i| = 1$. [3]



(b) $|z - 1 + i| \geq |z + 1 - i|$. [3]

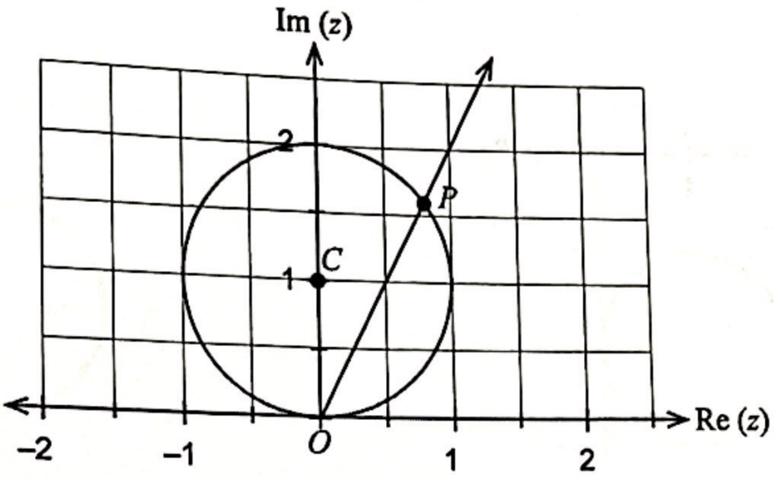


(c) For the locus $|z - 2i| = 1$ from part (a), state the exact maximum value for $|z + 2|$. [3]

2. **[6 marks]**

(MSPEC 2017:CF6)

A circle and a ray are indicated in the complex plane. The ray has equation $\arg(z) = \tan^{-1}(2)$. Point C is the centre of the circle. Point P is the intersection of the circle and the ray.



(a) Determine the equation for the circle. [2]

Point P determines a complex number $w = r\ cis\theta$.

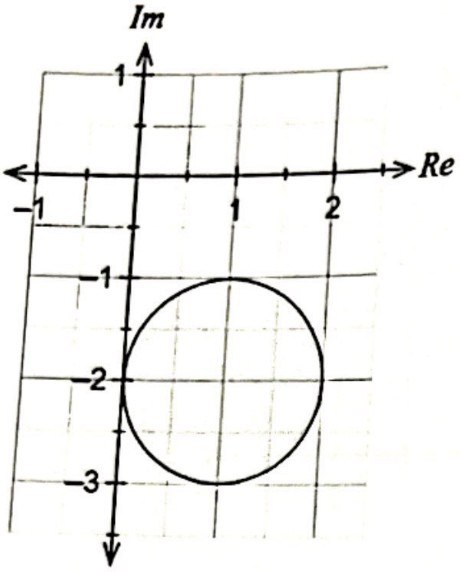
(b) Determine the exact values for r, θ. [4]

(MSPEC 2018:CA11)

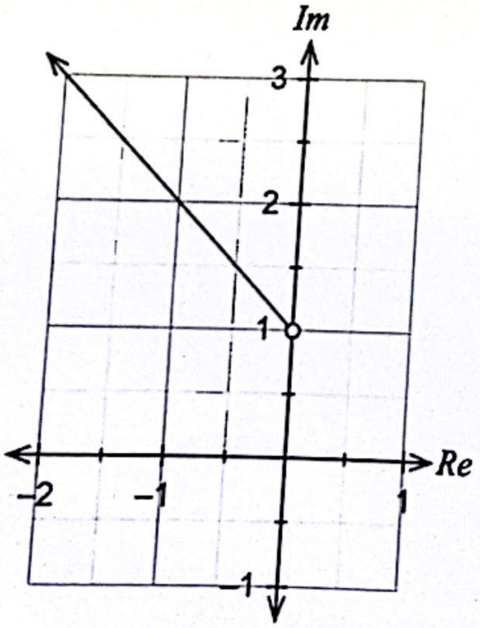
3. [8 marks]

A sketch of the locus of a complex number z is shown below. Write equations or inequalities in terms of z (without using $x = \text{Re}\,(z)$ or $y = \text{Im}\,(z)$) for each of the following:

(a) [3]

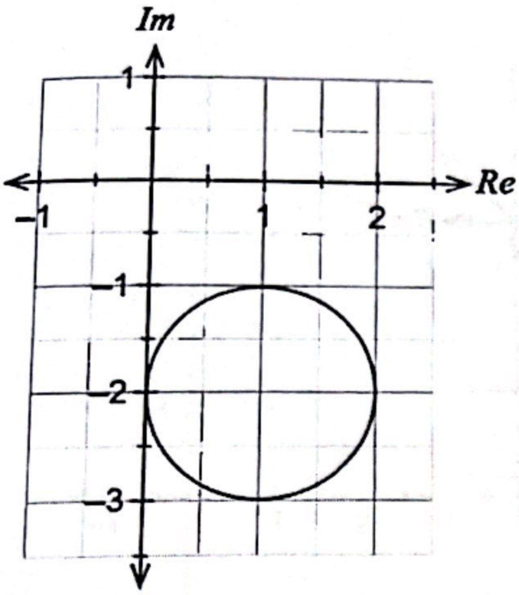


(b) [2]



3. (cont)

The sketch in part (a) from the previous page is repeated below, with only the circle indicated.

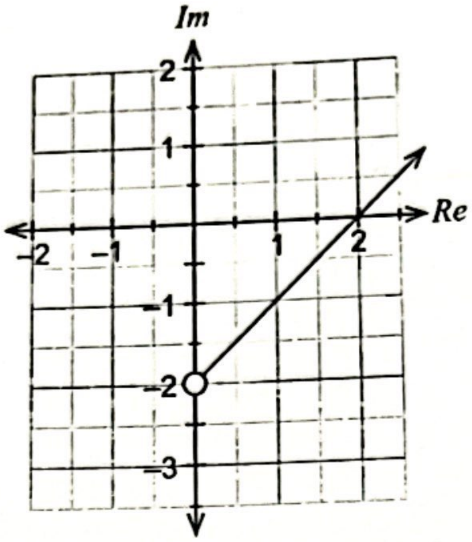


(c) For the locus above, determine the maximum value for arg (z) correct to 0.01, where $0 \le \arg(z) < 2\pi$. [3]

(MSPEC 2019:CA10)

4. **[5 marks]**

The sketch of the locus of a complex number $z = x + iy$ is shown below.

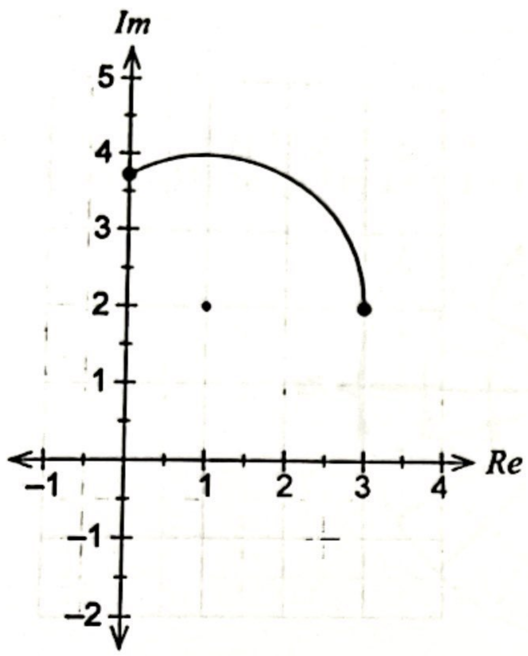


(a) Given that the equation for the above locus is written as $Arg(z - z_0) = k\pi$, determine the value of the constants z_0 and k. [2]

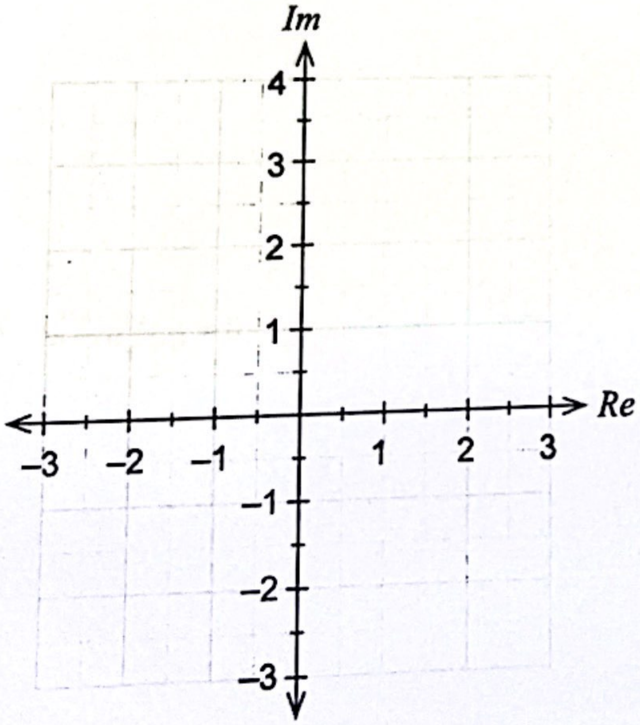
(b) Determine the minimum value for $|z-i|$ as an exact value. [3]

5. **[7 marks]** *(MSPEC 2020:CA10)*

(a) The sketch of the locus of a complex number z has been shown below. Write equations or inequalities in terms of z (without using $x = Re(z)$ or $y = Im(z)$) for the indicated locus. [4]



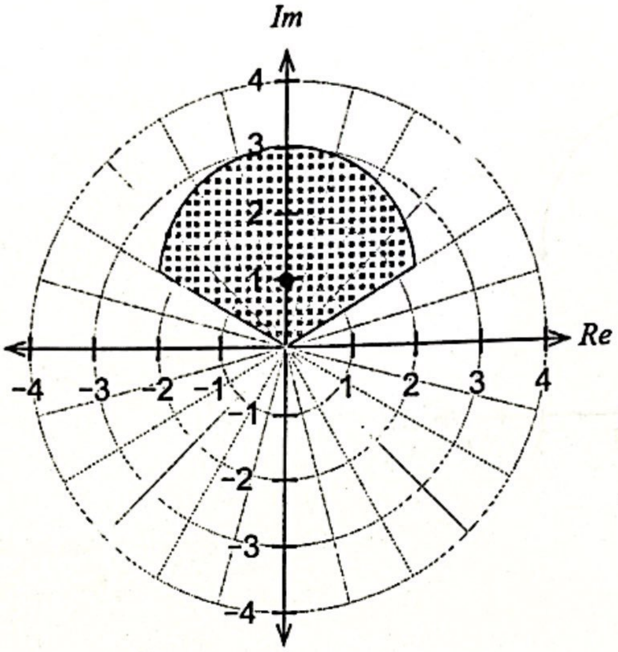
(b) Sketch the locus of the equation $|z+2| = |z-i| + \sqrt{5}$ in the Argand diagram below. [3]



(MSPEC 2021:CA11d)

6. **[4 marks]**

A sketch of the locus of a complex number z is shown below. The upper boundary of the locus is part of a circle, centred at $z = i$. Write equations or inequalities in terms of z (without using $x = Re(z)$ or $y = Im(z)$) for the indicated locus.



Solving Polynomial Equations

Chapter

4

1. **[7 marks]** *(Projected CF)*

(a) The function $f(z) = 2z^3 + az^2 + bz - 1$ has a factor of $2z - 1$ but a remainder of -6 is left when $f(z)$ is divided by $z + 1$. Find a and b.

(b) Solve $f(z) = 0$.

(MSPEC 2016:CF03)

2. **[5 marks]**

Consider $f(z) = z^3 + 2z^2 - 5z + 12$ where z is a complex number.

(a) Show that $(z + 4)$ is a factor of $f(z)$. [2]

(b) Solve the equation $z^3 + 2z^2 - 5z + 12 = 0$. [3]

3. **[6 marks]**

(MSPEC 2017:CF2)

Consider $f(z) = 2z^3 - 5z^2 + 4z - 10$ where z is a complex number.

(a) Show that $(z - \sqrt{2}i)$ is a factor of $f(z)$ [2]

(b) Given that $(z - \sqrt{2}i)$ is a factor of $f(z)$, state another factor of $f(z)$. [1]

(c) Solve the equation $2z^3 - 5z^2 + 4z - 10 = 0$. [3]

(MSPEC 2018:CF7)

4. **[6 marks]**

(a) Solve the equation $z^3 + 1 = 0$ giving solutions in polar form $r\ \mathrm{cis}\theta$. [3]

It can be shown that $P(z) = z^5 - 2z^4 + 5z^3 + z^2 - 2z + 5$ can be written in the form $P(z) = (z^3 + 1)\,Q(z)$.

(b) Determine $Q(z)$. [1]

(c) Hence solve the equation $z^5 - 2z^4 + 5z^3 + z^2 - 2z + 5 = 0$ giving all solutions in Cartesian form $a + bi$. [2]

5. **[6 marks]**

(MSPEC 2019:CF2)

Consider the function $P(z) = z^4 - 2z^3 + 14z^2 - 8z + 40$, defined over the complex numbers.

(a) Show that $(z - 2i)$ is a factor of $P(z)$. [2]

(b) Hence or otherwise, solve the equation $P(z) = 0$, giving solutions in the form $a + bi$. [4]

Composite and Inverse Functions

Chapter 5

1. **[11 marks]** *(Projected CF)*

Let $f(x) = 1 - x^2$ for $x \geq 0$ and $g(x) = \sqrt{1-x}$.

(a) State the domain and range of $g(x)$. [2]

(b) Determine expressions for $f(g(x))$ and $g(f(x))$. [2]

Note: $\sqrt{x^2} = |x|$ and $(\sqrt{x})^2 = x$.

(c) Determine the domain and range of $f(g(x))$. [2]

(d) Find $g^{-1}(x)$. [2]

(e) What is the relationship between $f(x)$ and $g(x)$? [1]

(f) Another function $h(x)$ is a one to one function and has a domain of $x \geq 2$ and range $y \leq 1$. State the domain and range of $h^{-1}(x)$. [2]

(MSPEC 2016:CF01)

2. **[4 marks]**

Note: This question requires knowledge of the ln(x) function which is typically not learnt till later in the year.

Functions f and g are defined as $f(x) = \ln(x)$ and $g(x) = \frac{1}{x}$.

(a) Determine an expression for $g \circ f(x)$. [1]

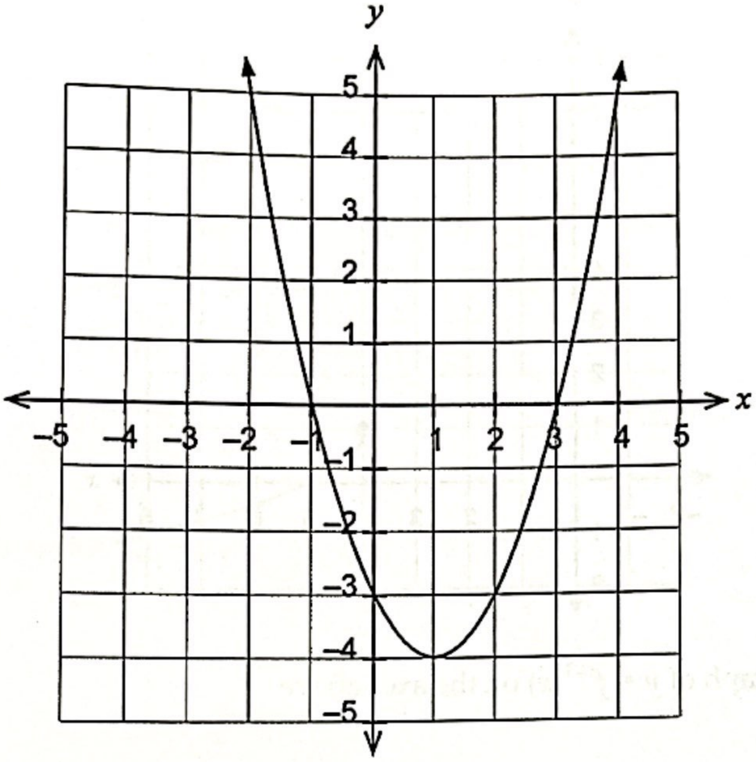
(b) For $g \circ f(x)$, state the:

(i) domain. [2]

(ii) range. [1]

3. **[5 marks]** *(MSPEC 2016:CF08c,d)*

The graph of $f(x) = (x - 1)^2 - 4$ is shown below.



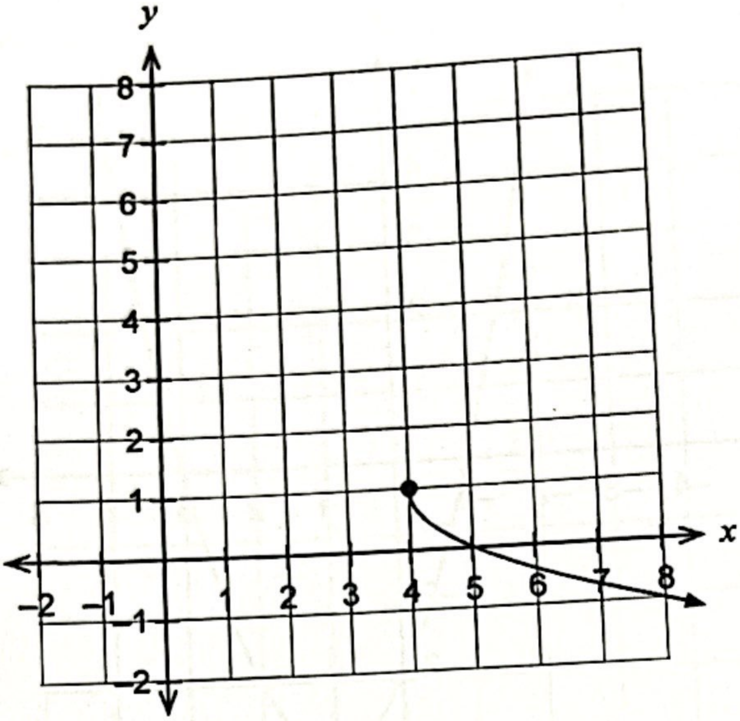
(a) The domain of function f is restricted to $x \leq k$ so that $y = f^{-1}(x)$ is a function. If this restricted domain represents the largest possible domain, state the value for the constant k. Explain. [2]

(b) Using the restriction $x \leq k$, determine the defining rule for $y = f^{-1}(x)$. Also state the domain for $y = f^{-1}(x)$. [3]

(MSPEC 2017:CF4)

4. **[9 marks]**

Function f is defined as $f(x) = 1 - \sqrt{x-4}$. The graph of $y = f(x)$ is shown below.



(a) Sketch the graph of $y = f^{-1}(x)$ on the axes above. [2]

(b) Determine the defining rule for $y = f^{-1}(x)$ and state its domain. [3]

4. (cont)

Function g is defined as $g(x) = \frac{1}{x^2}$.

(c) Determine an expression for $f \circ g(x)$. [1]

(d) For $f \circ g(x)$, determine the domain. [3]

(MSPEC 2018:CF1)

5. **[5 marks]**

Functions f, g are defined such that:

$$f(x) = \sqrt{x-3}$$

$$g(x) = \frac{x}{x-2}$$

(a) Determine $g \circ f(x)$. [1]

(b) Determine the domain for $g \circ f(x)$. [2]

(c) Given that $f^{-1}(x) = x^2 + 3$, is it true that $f^{-1}(-1) = 4$? [2]

Explain.

6. **[7 marks]**

(MSPEC 2019:CF4)

Functions f, g and h are defined such that:

$f(x) = \frac{1}{x-1}$, $g(x) = x^2$, $h(x) = \sqrt{x}$.

(a) Determine the defining rule for $f(h(x))$. [1]

(b) Determine the domain for $f(h(x))$. [2]

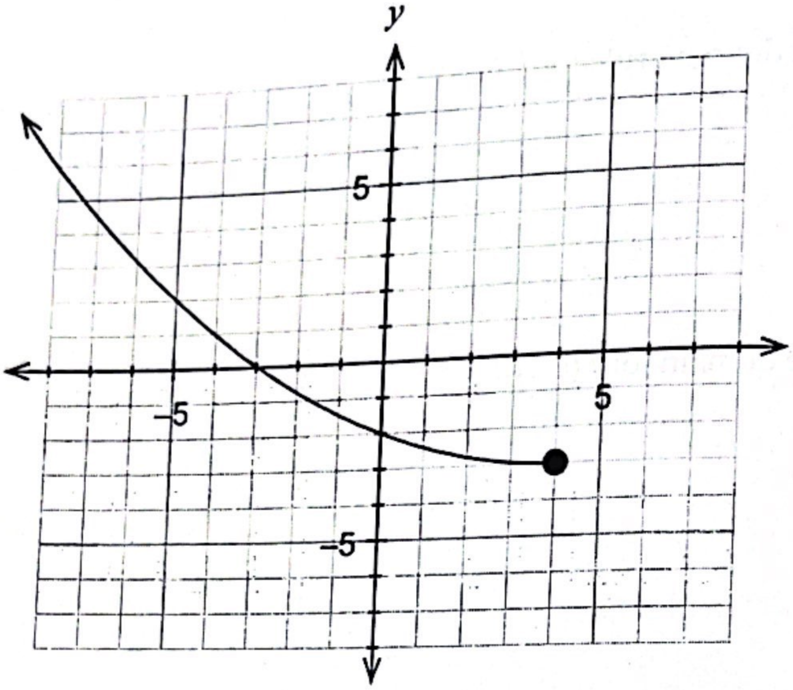
(c) Determine the range for $f(h(x))$. [2]

(d) Is it true that $f(h(g(x))) = \frac{1}{x-1} = f(x)$? Justify your answer. [2]

(MSPEC 2019:CF5)

7. **[6 marks]**

The graph of $y = g(x)$ is shown below.



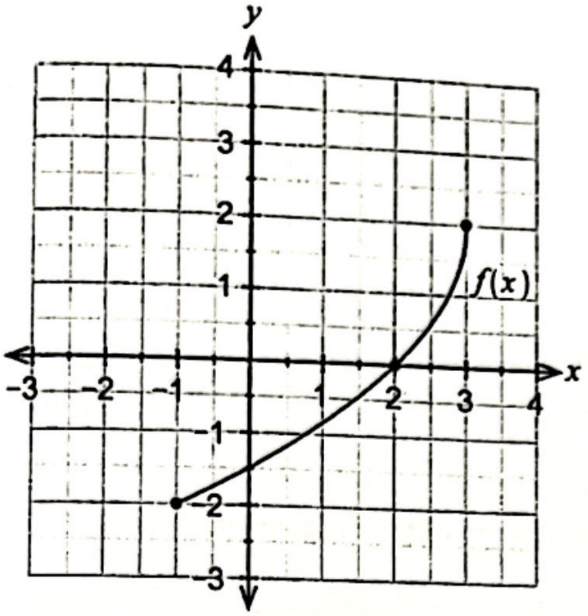
(a) Sketch the graph of $y = g^{-1}(x)$ on the axes above. [3]

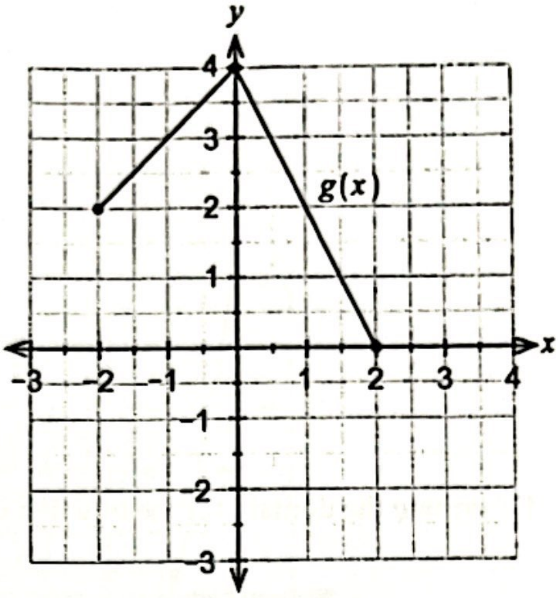
(b) Given that $g(x) = \frac{1}{16}(x-4)^2 - 3$ where $x \leq 4$, determine the defining rule for $y = g^{-1}(x)$. [3]

8. **[11 marks]**

(MSPEC 2021:CF02)

The graphs of functions f and g are shown below.





(a) Sketch the graph of function f^{-1} on the same axes used for function f. [2]

(b) Explain why the inverse of g is not a function. [1]

The defining rule for function f is $f(x) = 2 - 2\sqrt{3 - x}$ where $-1 \le x \le 3$.

(c) Determine the rule for $y = f^{-1}(x)$. [3]

8. (cont)

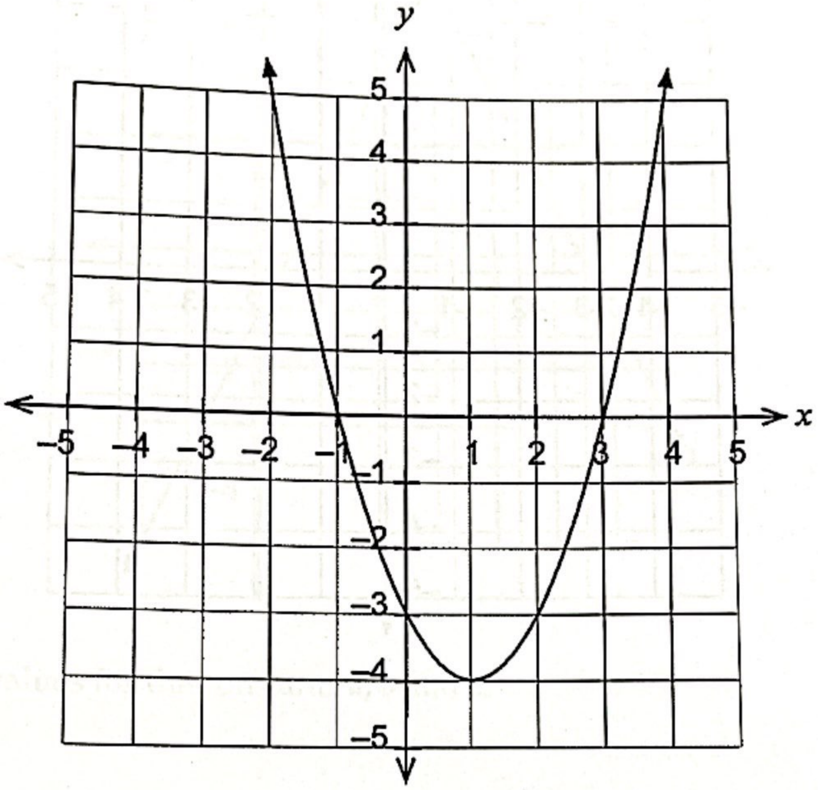
(d) Determine the exact value for $g(f(0))$. [2]

(e) Determine the domain for the function $y = f(g(x))$. Justify your answer. [3]

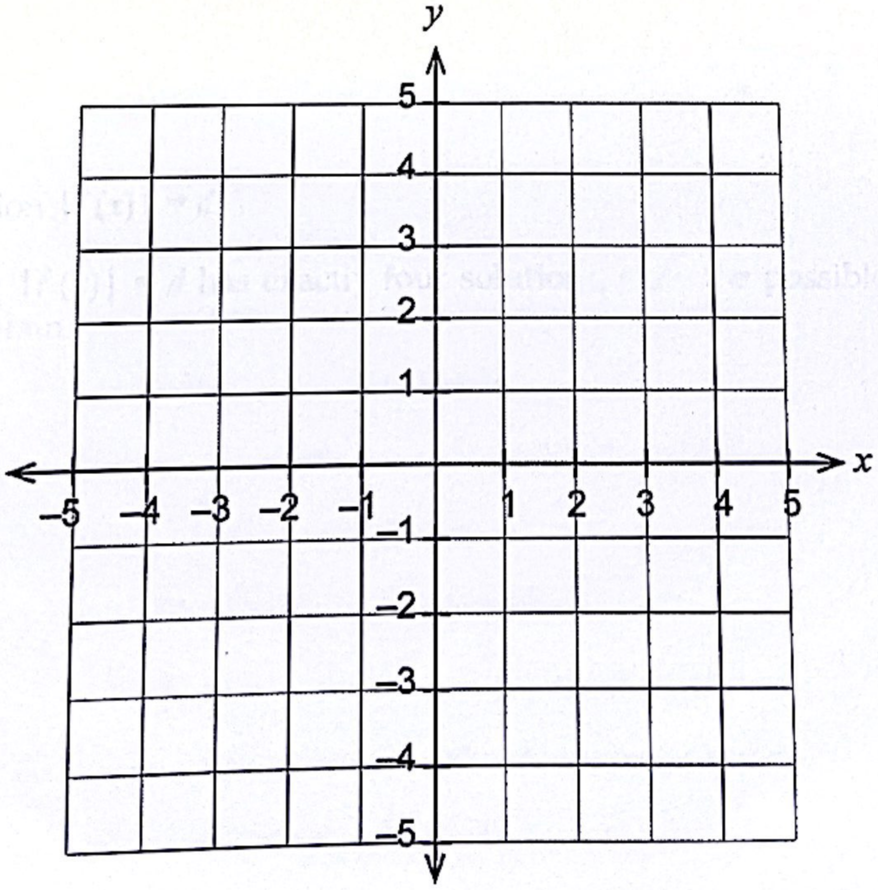
1. **[6 marks]**

(MSPEC 2016:CF08a,b)

The graph of $f(x) = (x-1)^2 - 4$ is shown below.

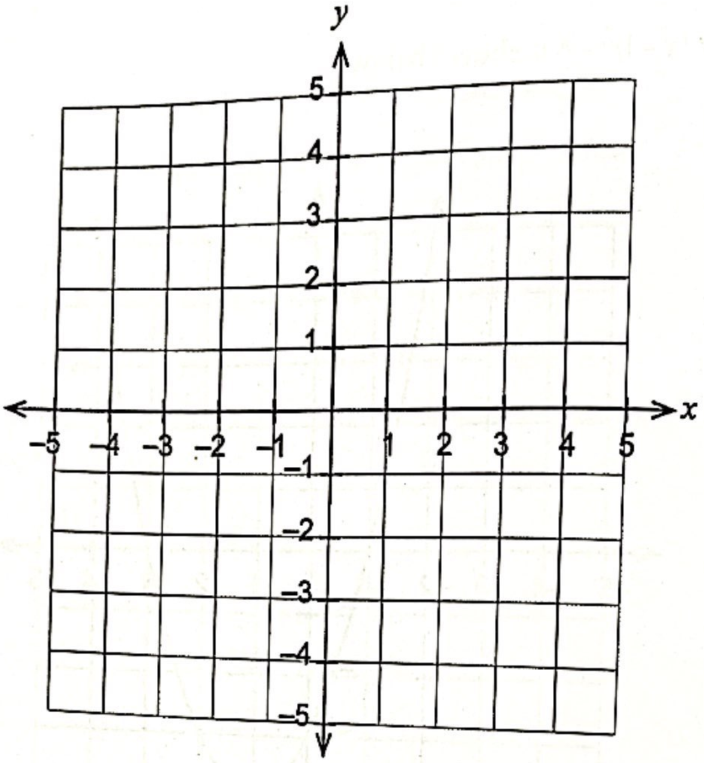


(a) Sketch the graph of $y = \dfrac{1}{f(x)}$ on the coordinate axes below. [4]



1. (cont)

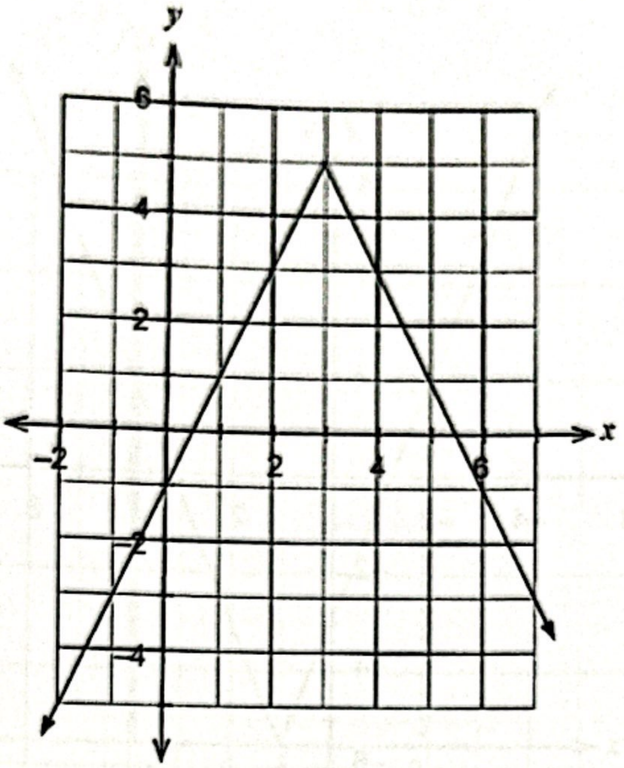
(b) Sketch the graph of $y = f(|x|)$ on the coordinate axes below. [2]



2. [6 marks]

(MSPEC 2016:CA12)

The graph of $f(x) = a\,|x - b| + c$ is shown below.



(a) Determine the values for the constants a, b and c. [3]

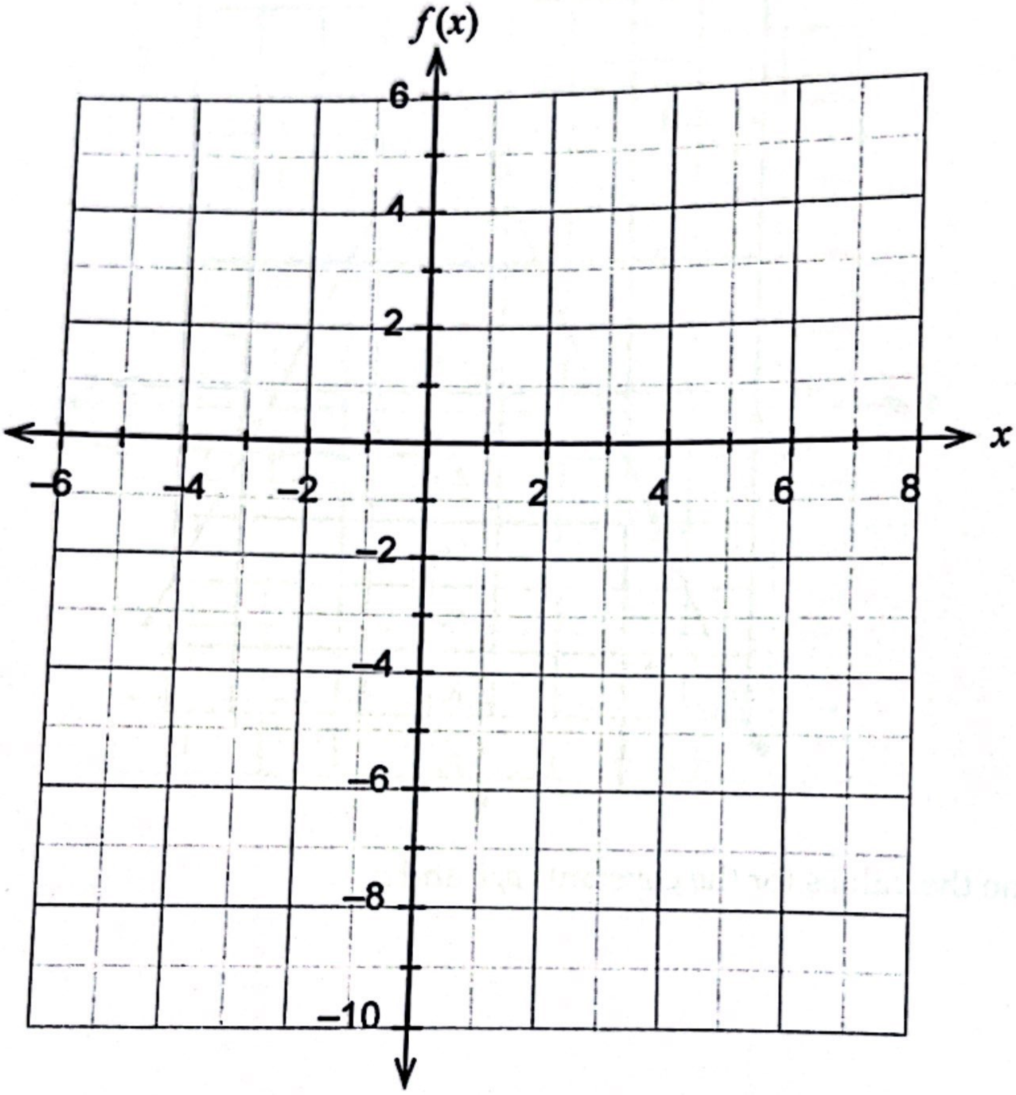
Consider the equation $|f(x)| = d$.

(b) If the equation $|f(x)| = d$ has exactly four solutions, state the possible value(s) for the constant d. Explain. [3]

(MSPEC 2017:CF5)

3. **[6 marks]**

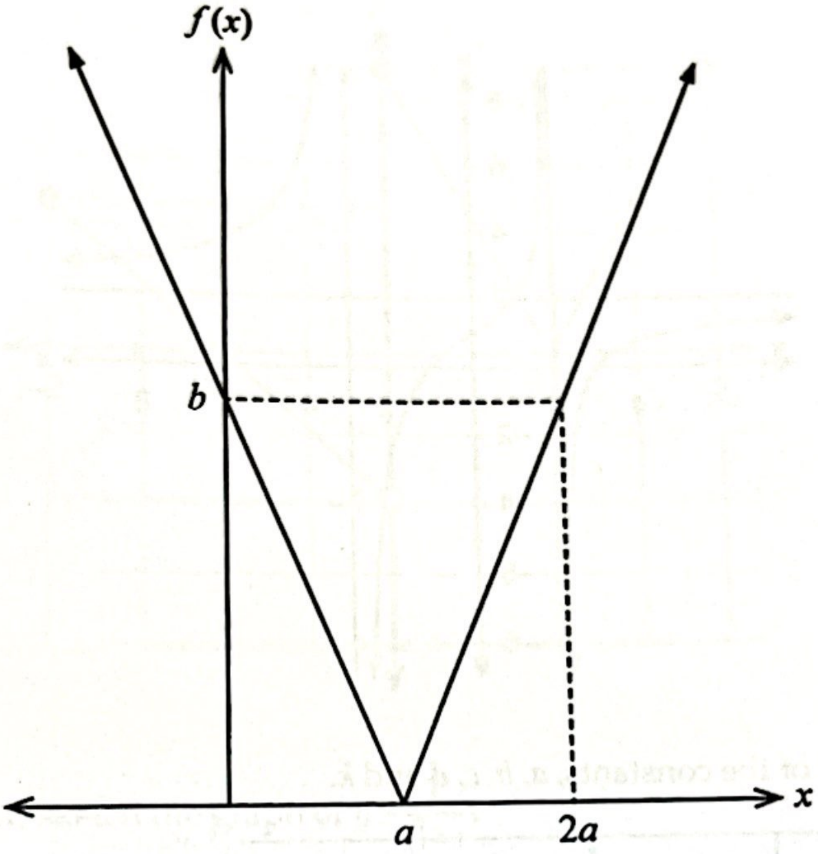
Sketch the graph of $f(x) = -\dfrac{4(x-3)(x+1)}{x^2-2x-8}$ on the axes below.



4. **[3 marks]**

(MSPEC 2017:CA16*a*)

Function f is defined by its graph shown below. The constants a, $b > 0$ where $b > a$.

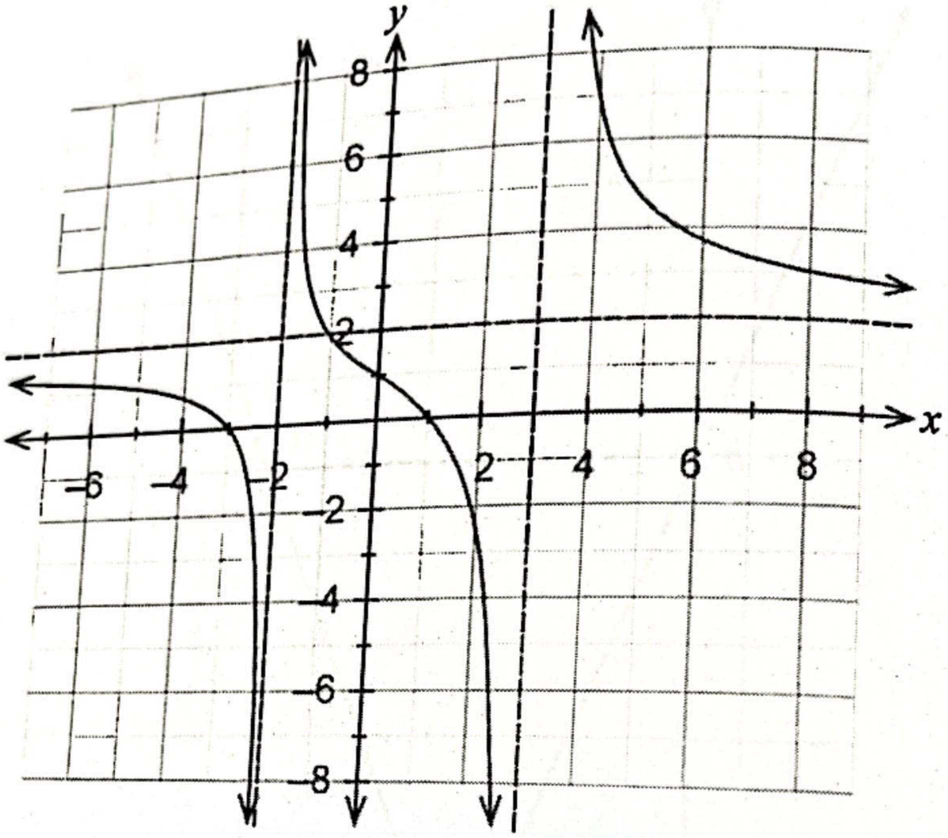


Determine the defining rule for function $f(x)$ in terms of a, b.

(MSPEC 2

5. **[4 marks]**

The graph of $f(x) = \dfrac{k(x-a)(x-b)}{(x-c)(x-d)}$ is shown below.



Determine the value of the constants, a, b, c, d and k.

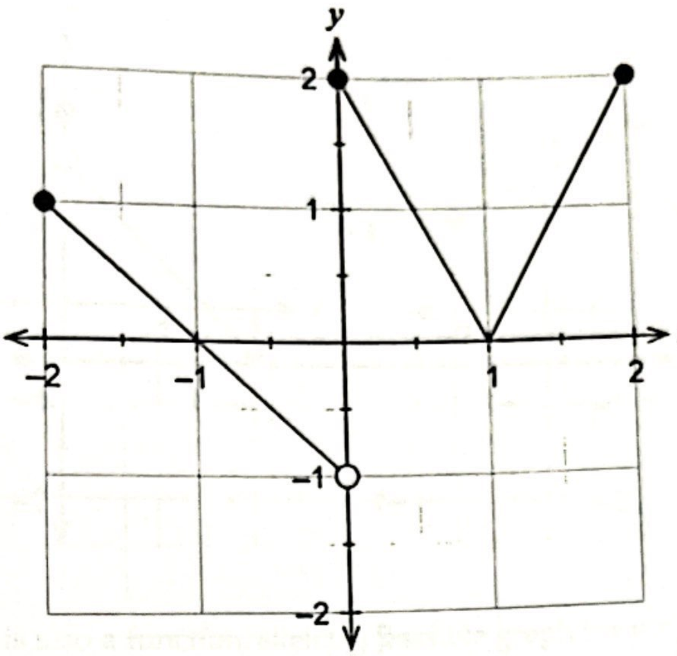
a	b	c	d	k

Explain your choice for the value of k.

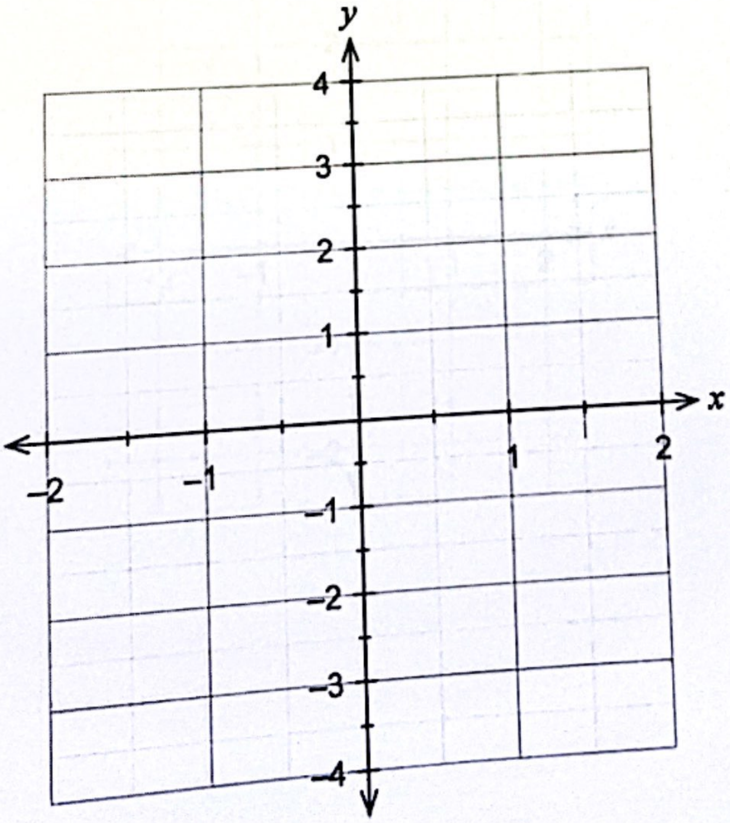
6. **[9 marks]**

(MSPEC 2018:CA14)

The graph of $y = f(x)$ is shown below.

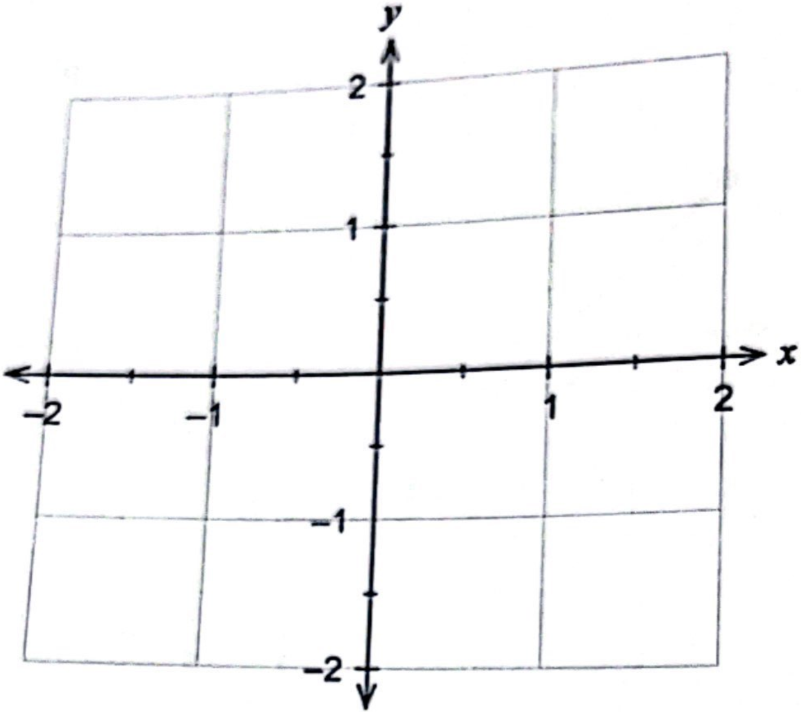


(a) On the axes below, sketch the graph of $y = \frac{1}{f(x)}$. [4]



6. (cont)

(b) On the axes below, sketch the graph of $y = f(-|x|)$. [3]

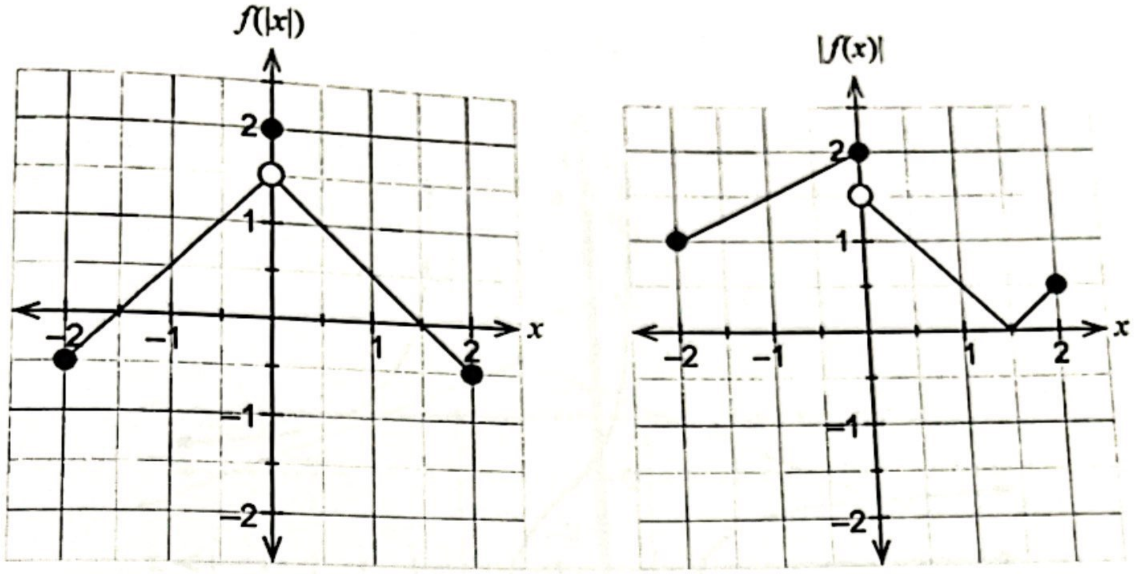


(c) Solve the equation $|f(x) - 1| = 1$. [2]

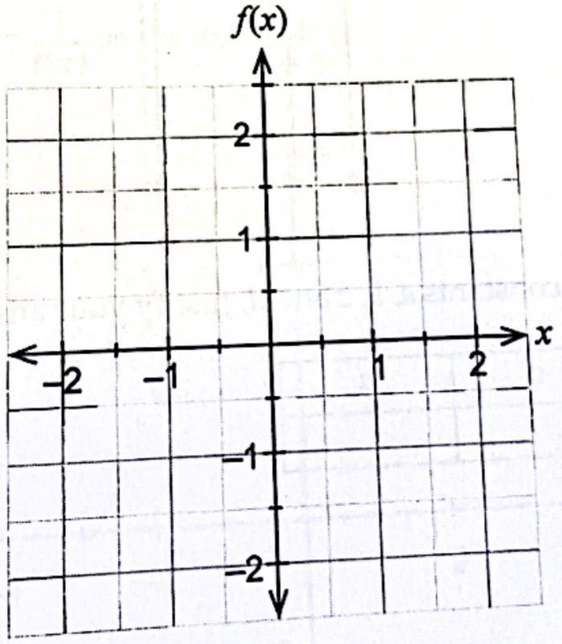
7. **[4 marks]**

(MSPEC 2019:CF7)

The graphs of $y = f(|x|)$ and $y = |f(x)|$ are shown below.



Given that $y = f^{-1}(x)$ is also a function, sketch a possible graph for $y = f(x)$ on the axes below. Justify your answer considering $y = f^{-1}(x)$.

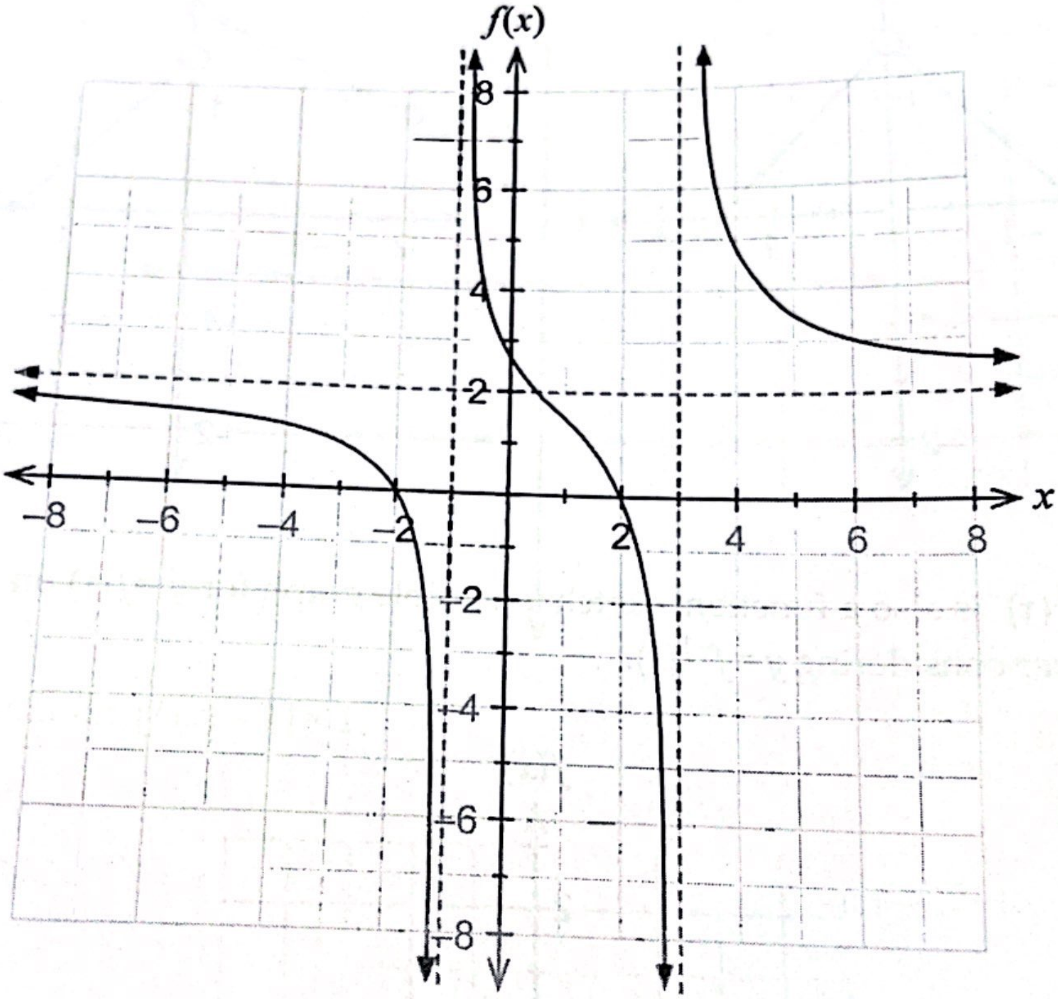


(MSPEC 2020

8. **[6 marks]**

The graph of $y = f(x)$ is shown on the axes below. The defining rule is given by

$f(x) = \dfrac{a(x^2 - b)}{(x + c)(x - d)}$ where a, b, c and d are positive constants.



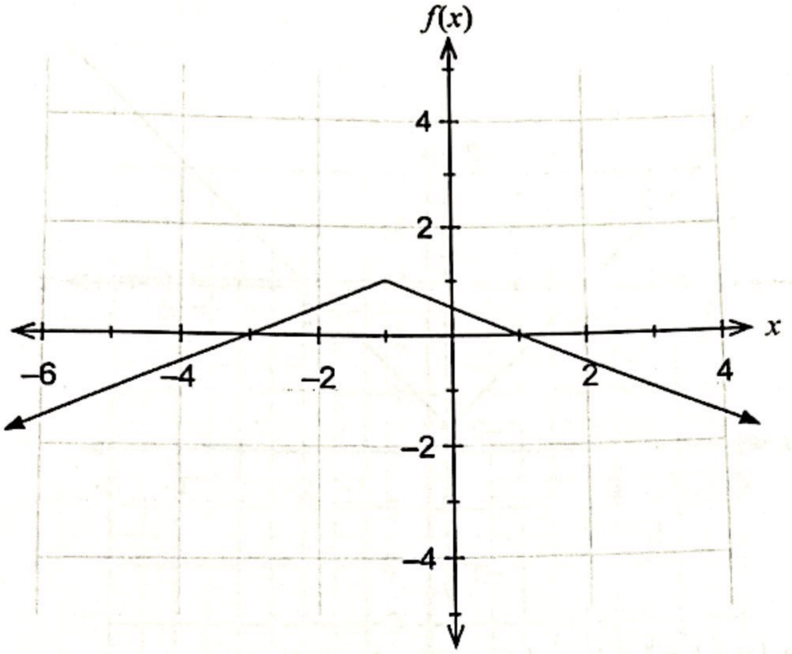
Determine the value of the constants a, b, c and d. Justify your answers.

a	b	c	d

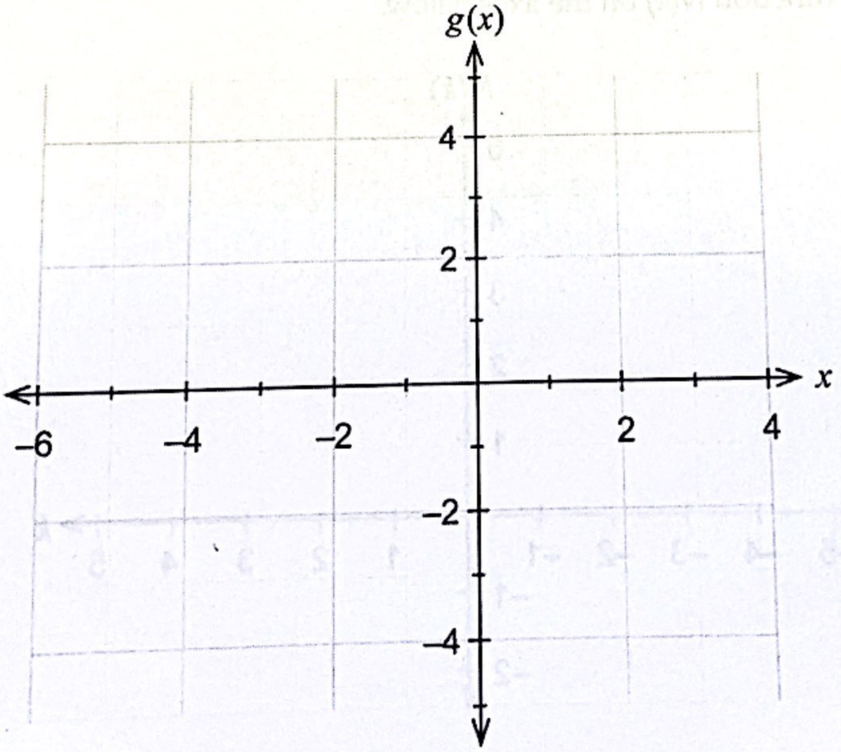
9. **[7 marks]**

(MSPEC 2020:CF5)

The graph of $f(x) = 1 - \frac{|x+1|}{2}$ is shown below.



(a) Sketch the graph of $g(x) = \frac{1}{f(x)}$ on the axes below. [4]

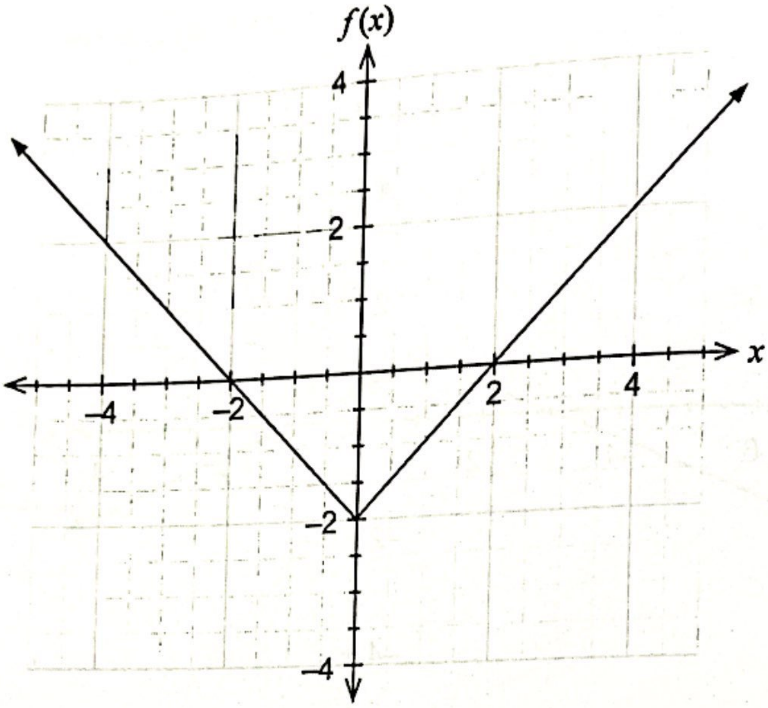


(b) Hence give the domain and range for $h(x) = \frac{4}{2-|x+1|}$. [3]

(MSPEC 2020:CA21)

10. **[5 marks]**

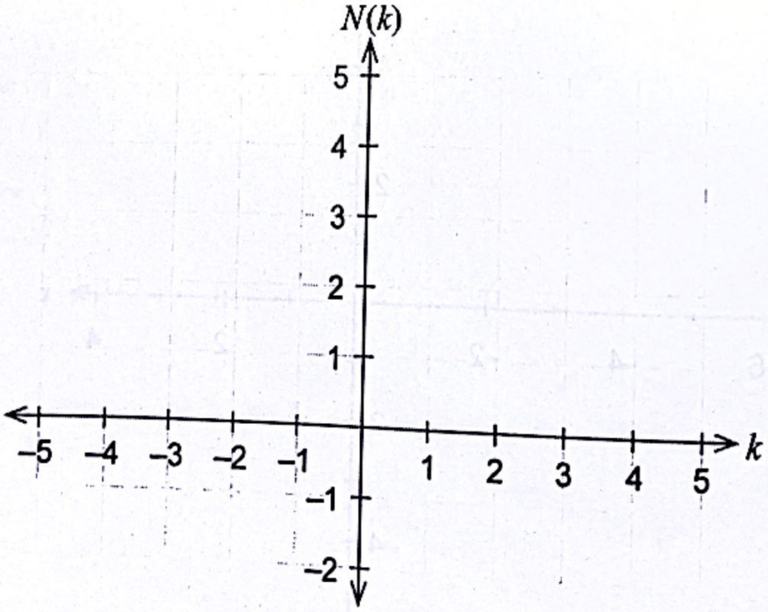
The sketch of the graph of $y = f(x)$ is shown below.



Consider the equation $|f(x)| = k$ where k is any real constant.

Define function $N(k)$ = the number of real solutions to the equation $|f(x)| = k$.

Sketch the graph of function $N(k)$ on the axes below.

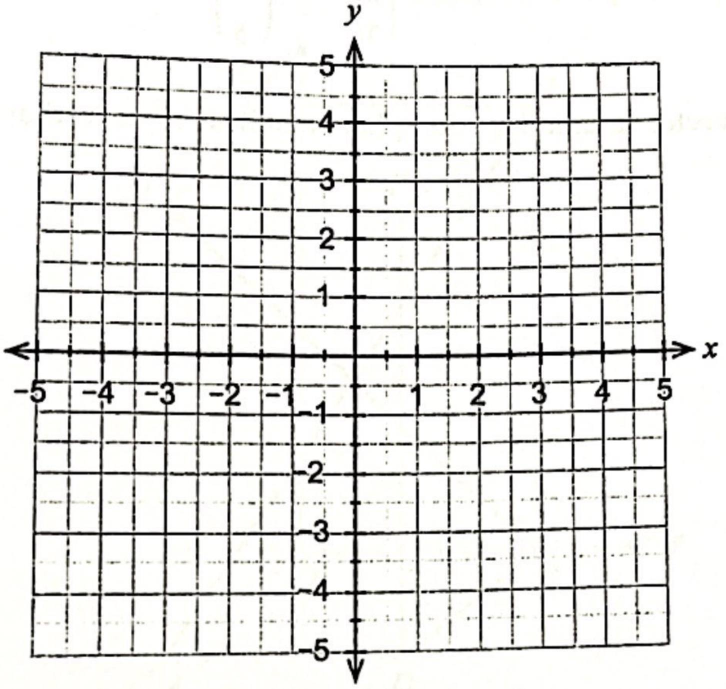


11. **[5 marks]**

(MSPEC 2021:CF04)

Consider the function $f(x) = \dfrac{x^2 - 4}{x + 1} = x - 1 - \dfrac{3}{x + 1}$.

Sketch the graph of the function $y = f(x)$ on the axes below. Indicate clearly the x and y intercepts and any asymptotes.



Chapter

7

Vectors

(MSPEC 2016:CF07)

1. **[7 marks]**

Points A, B have respective position vectors $\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix}$.

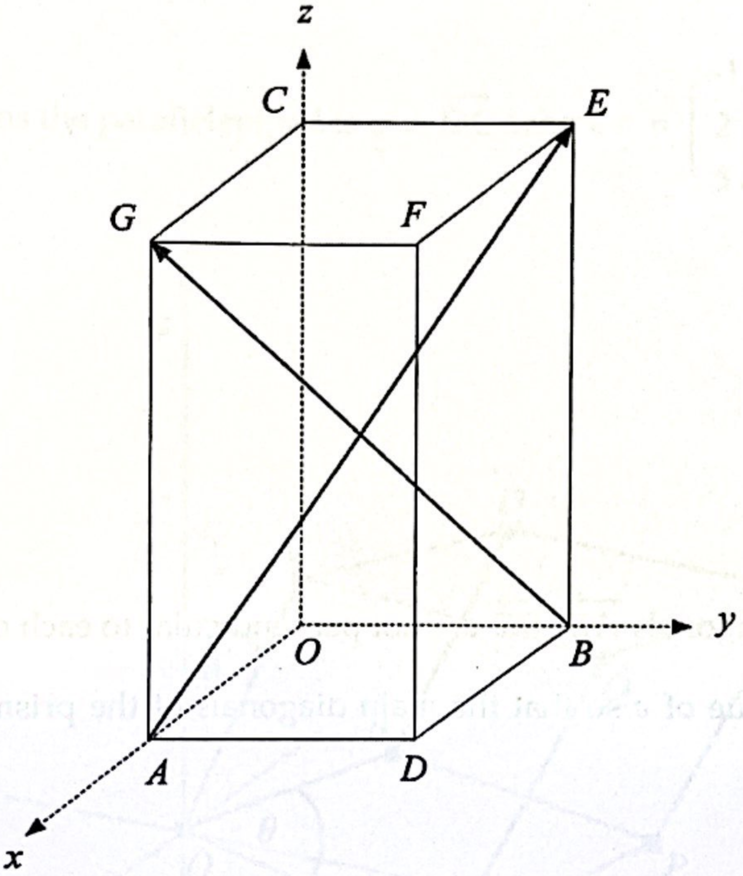
(a) Determine the vector equation for the sphere that has $\overline{AB}$ as its diameter. [3]

If the point O is the origin, consider the plane that contains the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

(b) Determine the vector equation for this plane in the form $\underset{\sim}{r} \cdot \underset{\sim}{n} = c$. [4]

2. **[10 marks]** *(MSPEC 2017:CF7)*

A right rectangular prism, with square base $OADB$, is shown below. Point O is the origin and points A, B, C have respective position vectors $\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$ where $c > 0$.



(a) Determine, in terms of c, the:

(i) vector equation for the line containing points A and E. [3]

2. (cont)

(ii) Cartesian equation for the plane $ADEC$. [4]

In general, the main diagonals $\overrightarrow{AE}$, $\overrightarrow{BG}$ are not perpendicular to each other.

(b) Determine the value of c so that the main diagonals of the prism are perpendicular to each other. [3]

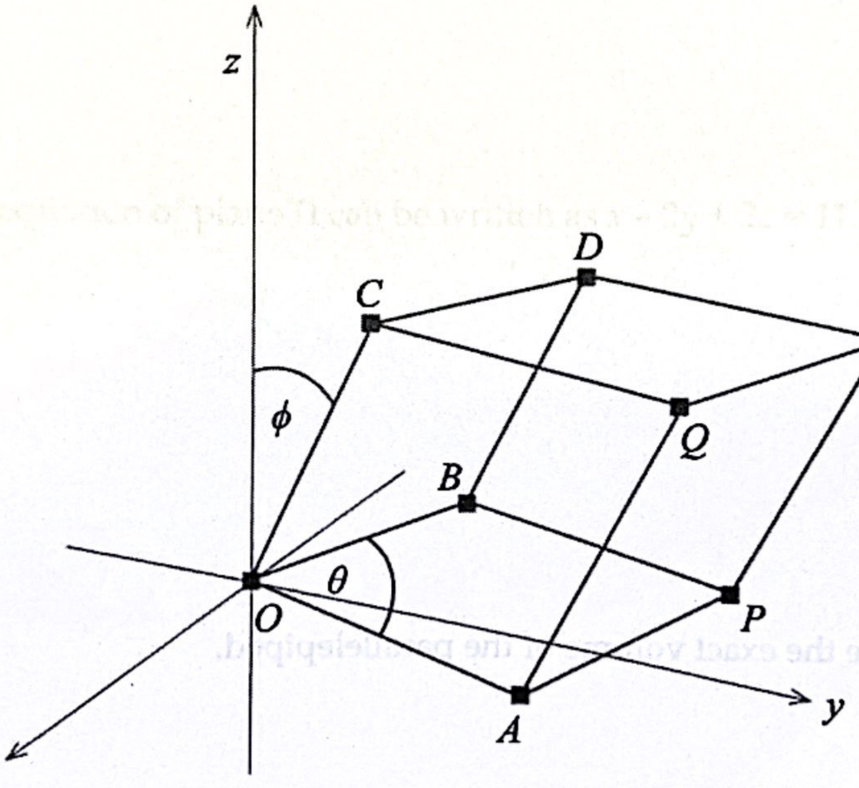
3. **[5 marks]**

(MSPEC 2018:CF8)

A parallelepiped is a prism where each face is a parallelogram. Let $OAPB$ be the parallelogram formed by the horizontal sides $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ where

$$\underset{\sim}{a} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} \text{ and } \underset{\sim}{b} = \begin{pmatrix} -8 \\ 2 \\ 0 \end{pmatrix}.$$

The third side that forms the parallelepiped is $\underset{\sim}{c} = \overrightarrow{OC}$ where $\underset{\sim}{c} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$.



Let θ = the size of $\angle AOB$

ϕ = the angle between $\overrightarrow{OC}$ and the positive z axis

(a) Determine $\underset{\sim}{a} \times \underset{\sim}{b}$. [2]

3. (cont)

The volume of any prism can be found by considering the formula $Volume = Area\ (Base) \times h$, where h = the perpendicular height of the prism.

It is also true that $\left|\underset{\sim}{a} \times \underset{\sim}{b}\right| = \left|\underset{\sim}{a}\right| \left|\underset{\sim}{b}\right| \sin\theta$.

(b) Explain why $\underset{\sim}{c} \cdot (\underset{\sim}{a} \times \underset{\sim}{b})$ will determine the volume of the parallelepiped. [2]

(c) Hence determine the exact volume of the parallelepiped. [1]

4. **[7 marks]** *(MSPEC 2018:CA17)*

Plane Π is represented by the equation: $\underset{\sim}{r} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

(a) Determine $\begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and describe what this represents. [1]

(b) Show that the equation of plane Π can be written as $x - 2y + 2z = 11$. [2]

Consider sphere S with its centre at point A (3, 4, −1).

(c) Determine the Cartesian equation for S if plane Π is tangential to this sphere. [4]

(MSPEC 2019:CA16)

5. **[12 marks]**

Plane Π_1 has Cartesian equation $z = 2x + y + 4$.

(a) Determine a vector that is normal to plane Π_1. [2]

Line L has equation $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

(b) Determine the point of intersection between line L and plane Π_1. [3]

5. (cont)

Plane Π_2 contains line L and is perpendicular to plane Π_1.

(c) Determine the vector equation for plane II_2. [4]

Sphere S has vector equation $\left|\underset{\sim}{r} - (3\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k})\right| = \sqrt{35}$.

(d) Determine whether line L is a tangent to sphere S. Justify your answer. [3]

(MSPEC 2019:CA19)

6. **[4 marks]**

Two parallel planes Π_1 and Π_2 have their equations given by:

$\Pi_1 \quad \underset{\sim}{r} \bullet \underset{\sim}{n} = 11$

$\Pi_2 \quad \underset{\sim}{r} \bullet \underset{\sim}{n} = -4 \qquad$ where $\underset{\sim}{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

It is known that (2,3,–7) is a point on plane Π_1.

Prove the distance d between the point (2,3,–7) and plane Π_2 is given by $d = \dfrac{15}{\sqrt{a^2 + b^2 + c^2}}$.

7. **[5 marks]**

(MSPEC 2020:CF2)

Plane II has vector equation $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

(a) Determine the normal vector $\underset{\sim}{n}$ for plane II. [3]

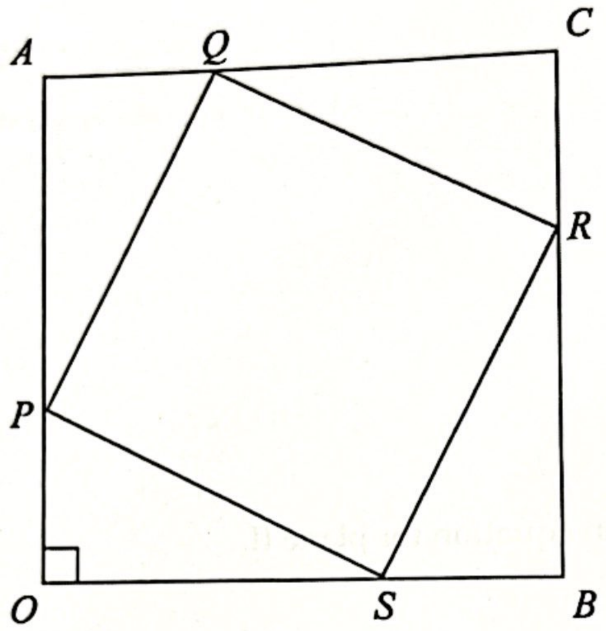
(b) Determine the Cartesian equation for plane II. [2]

(MSPEC 2020:CA20a)

8. **[5 marks]**

Consider square $OACB$ where point O is the origin. Let the position vectors for points A, B be defined as $\underset{\sim}{a}$, $\underset{\sim}{b}$ respectively i.e. $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.

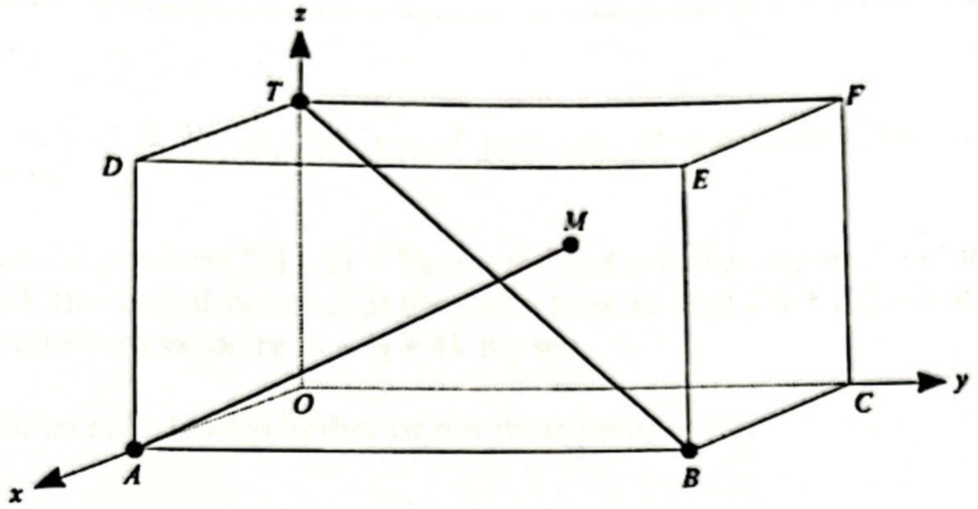
Let points P, Q, R and S be defined so that $\overrightarrow{OP} = k\underset{\sim}{a}$, $\overrightarrow{AQ} = k\underset{\sim}{b}$, $\overrightarrow{RC} = k\underset{\sim}{a}$ and $\overrightarrow{SB} = k\underset{\sim}{b}$ where $0 \le k \le 1$. This means that points P, Q, R and S are positioned along their respective sides in equal proportion.



Using vector methods, prove that the size of $\angle PQR = 90°$.

9. **[8 marks]** *(MSPEC 2021:CA16)*

A rectangular prism is defined using the coordinate system shown with $A\ (2, 0, 0)$, $C\ (0, 4, 0)$ and $T\ (0, 0, 3)$. Point M is the centre of the planar face $OCFT$ with coordinates $(0, 2, 1.5)$.



(a) Determine the vector equation for the prism's main diagonal $\overleftrightarrow{BT}$. [2]

9. (cont)

(b) Determine the Cartesian equation of the sphere that contains all vertices of the rectangular prism. [3]

(c) Prove, using a vector method, that line $\overleftrightarrow{AM}$ does **not** intersect $\overleftrightarrow{BT}$. [3]

Applications of Vectors in Three Dimensions

Chapter

8

1. **[4 marks]** *(Projected CA)*

Two particles, A and B, leave their initial positions at the same time and travel with constant velocities.

Particle A has initial position 10**i** – 5**j** + 5**k** metres and is travelling with velocity 3**i** + **j** – 2**k** m/sec. Particle B has initial position at the same time as A at 28**i** + 22**j** – 31**k** metres and is travelling with constant velocity 2**i** – 4**j** + 4**k** m/sec.

Find if the particles collide or whether or not their paths cross.

2. **[7 marks]** *(3CDMAS 2015:CA18)*

Archie and Brianna are two radio-controlled drones carrying cameras. Archie leaves the point with position vector 2i – 3j + 5k kilometres at 2 pm, and travels with a constant velocity of 10i – 10j – 5k kilometres per hour.

(a) Find the speed at which Archie moves. [1]

(b) Calculate the position vector of the location of Archie at 4 pm. [1]

The second radio-controlled drone, Brianna, leaves the point with position vector –20i + 10j – 30k kilometres at 4 pm. Brianna travels with a constant velocity of 30i – 30j – 15k kilometres per hour.

(c) Calculate the position vector of the location of Brianna at any time *t*. [1]

2. (cont)

(d) Find an expression for the vector representing the distance between Archie and Brianna at any time t. [1]

(e) Use calculus to determine the shortest distance between Archie and Brianna. [2]

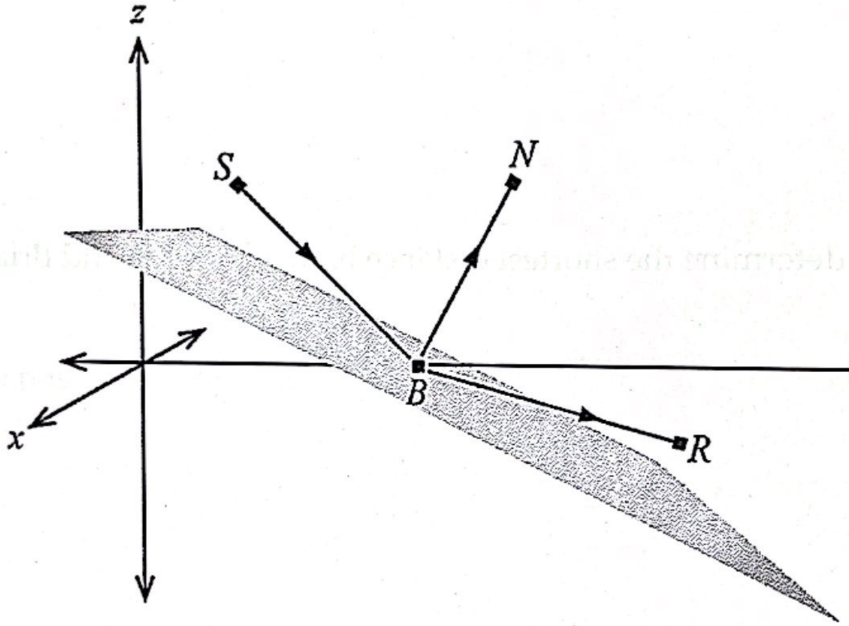
(f) Correct to the nearest minute, at what time are they closest? [1]

(MSPEC 2016:CA20)

3. **[7 marks]**

A laser pointer at point S directs a highly focused beam of light towards a mirror. The beam bounces off the mirror at point B and is then reflected away from the mirror toward point R.

The mirror's surface is given by the equation $\underset{\sim}{r} \cdot (\underset{\sim}{j} + 2\underset{\sim}{k}) = 9$ and the laser pointer is positioned at point S with position vector $-2\underset{\sim}{i} + 3\underset{\sim}{j} + 6\underset{\sim}{k}$. The laser pointer is held so that the beam is pointed in the direction $\underset{\sim}{d}_1 = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$.



(a) Determine the position vector for point B. [4]

3. (cont)

The laser beam is reflected away from the mirror so that:

- the angle of the incoming beam $\overrightarrow{SB}$ to the normal of the mirror is equal to the angle of the reflected beam $\overrightarrow{BR}$ to the normal of the mirror i.e. $s\angle SBN = s\angle NBR$.

- the incoming beam $\overrightarrow{SB}$, the normal of the mirror and the reflected beam $\overrightarrow{BR}$ are all contained in one plane.

Let $\underset{\sim}{\hat{d}}_2$ = the unit vector in the direction of the reflected beam $\overrightarrow{BR}$ i.e. $|\underset{\sim}{\hat{d}}_2| = 1$.

(b) Determine the unit vector $\underset{\sim}{\hat{d}}_2$, giving components correct to 0.01. [3]

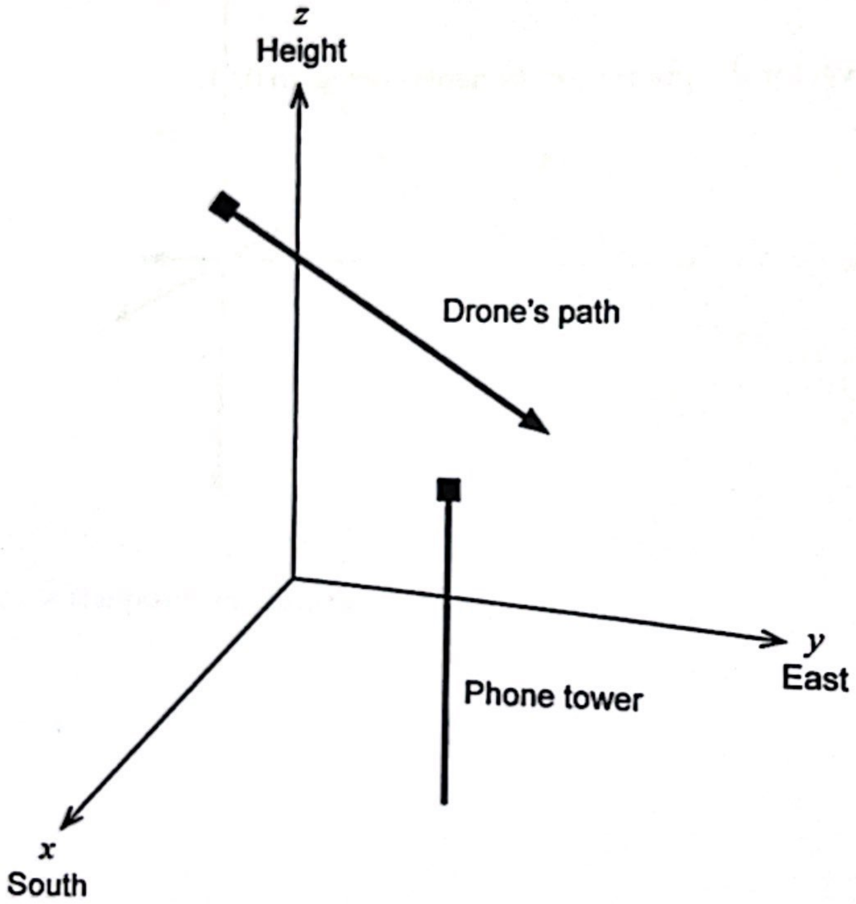
(MSPEC 2017:CA14)

4. [7 marks]

A small drone is launched and, after hovering in an initial position, it flies in a straight line under the control of its operator. The position of the drone from the operator is given by

$\underset{\sim}{r}(t) = \begin{pmatrix} 100 + 0.5t \\ 0.6t \\ 50 - 0.02t \end{pmatrix}$ metres, where t is the time in seconds it has been flying in a straight line.

The top of a mobile phone tower is positioned at $200\underset{\sim}{i} + 150\underset{\sim}{j} + 30\underset{\sim}{k}$ relative to the operator i.e. the mobile phone tower is 30 metres tall.



(a) After two minutes of flight, how high is the drone above the ground? [2]

4. (cont)

(b) Write the expression for the position vector of the drone from the top of the phone tower after *t* seconds. [1]

The operator knows that the drone will not strike the mobile phone tower. However, the operator does not know that the drone will cause interference when it is less than 50 metres from the top of the tower.

(c) Determine whether the drone will cause interference to the mobile phone tower and, if so, for how long will this occur, correct to the nearest second. [4]

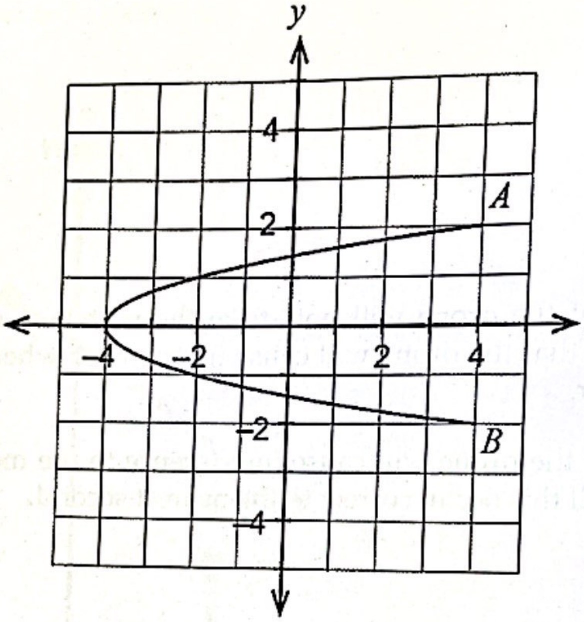
Chapter 9

Vector Calculus in Two Dimensions

(MSPEC 2016:CA16)

1. **[10 marks]**

A particle's position vector $\underset{\sim}{r}(t)$ is given by $\underset{\sim}{r}(t) = \begin{pmatrix} 4\cos 2t \\ 2\cos t \end{pmatrix}$ centimetres where t is measured in seconds. A plot of the path of the particle is shown below.



(a) Express the path of the particle as a Cartesian equation. [3]

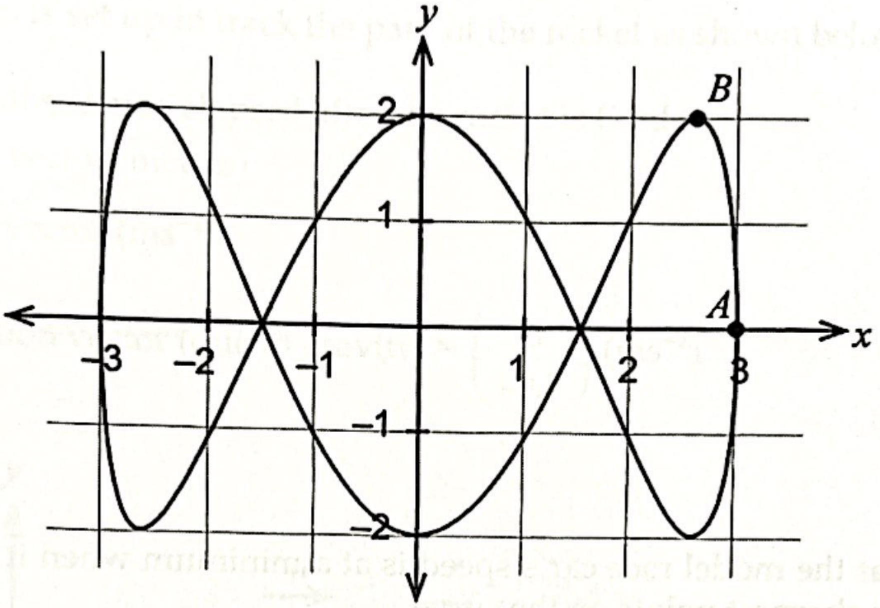
(b) Determine the speed of the particle, correct to 0.01 cm per second, when it first reaches the point where $x = -2$. [4]

(c) Write the expression, in terms of trigonometric functions, for the distance the particle will travel along its path in travelling from point A to point B. Do **not** evaluate this expression. [3]

2. **[13 marks]**

(MSPEC 2017:CA15)

A battery-powered model race car moves around a race track as indicated in the diagram below. The car's initial position is point *A*.



At any time *t* seconds, the velocity vector $\underset{\sim}{v}(t)$ of the model race car is given by:

$$\underset{\sim}{v}(t) = \begin{pmatrix} -\sin\left(\frac{t}{3}\right) \\ 2\cos(t) \end{pmatrix} \text{ metres per second.}$$

(a) Determine the initial velocity vector and show this on the diagram above. [2]

(b) Write an expression that will determine the change in displacement over the first $\frac{3\pi}{2}$ seconds.

[2]

2. (cont)

(c) Determine the displacement vector $\underset{\sim}{r}(t)$. [3]

It can be shown that the model race car's speed is at a minimum when it reaches point *B* on the track, one of the sharpest points on the curve.

(d) Determine the acceleration vector $\underset{\sim}{a}$ when the car reaches point *B*, giving components correct to 0.01. [3]

(e) Determine the distance, correct to 0.01 metres, that the model race car travels in completing one lap of the track. [3]

3. **[9 marks]**

(MSPEC 2018:CA19)

A small rocket is fired from the ground at an angle of $\theta°$ to the horizontal with a speed of 70 metres per second. The rocket has the assistance of a steady wind that is blowing horizontally at w metres per second.

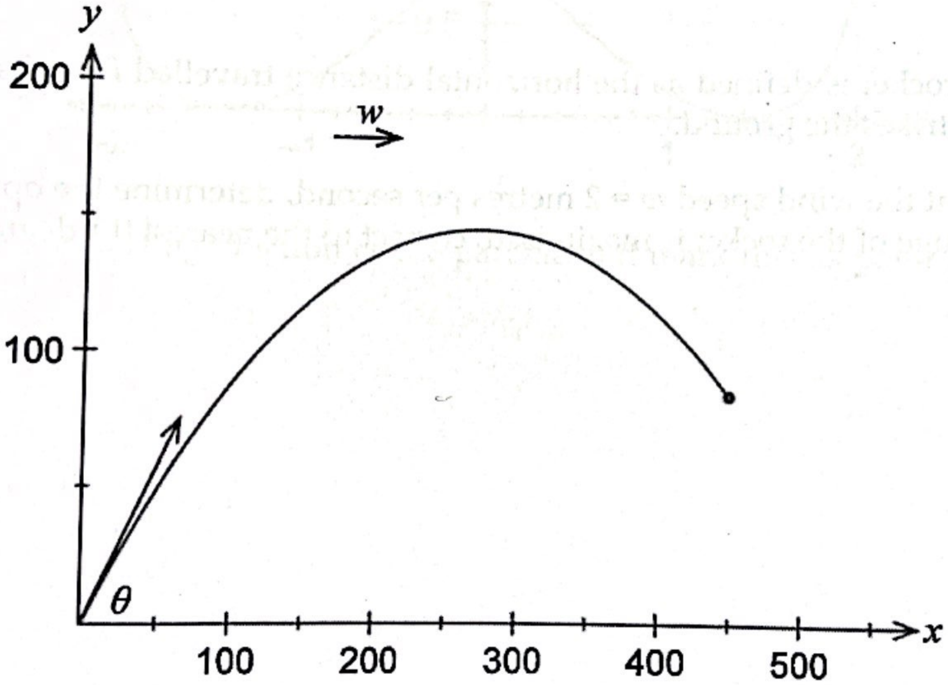
A coordinate system is set up to track the path of the rocket as shown below.

Let t = the number of seconds elapsed after the rocket is fired

$\underset{\sim}{r}(t)$ = the position vector (metres)

$\underset{\sim}{v}(t)$ = the velocity vector (ms^{-1})

$\underset{\sim}{a}(t)$ = the acceleration vector (due to gravity) = $\begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$ (ms^{-2})



(a) Given $\underset{\sim}{a}(t) = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$, show that $\underset{\sim}{r}(t) = \begin{pmatrix} (70\cos\theta + w)t \\ (70\sin\theta)t - 4.9t^2 \end{pmatrix}$. [3]

3. (cont)

(b) Obtain the Cartesian equation for the path of the rocket, in terms of θ and w. [2]

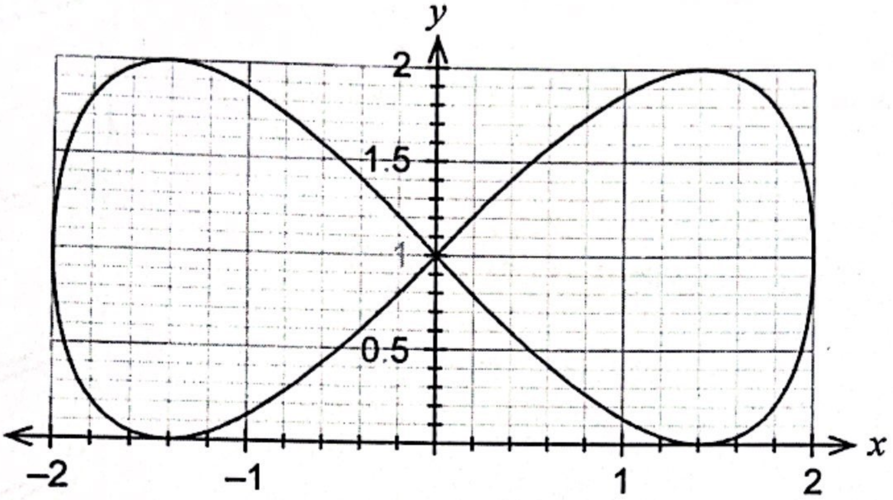
The range of the rocket is defined as the horizontal distance travelled from its launch to the point at which it strikes the ground.

(c) Assuming that the wind speed $w = 2$ metres per second, determine the optimum angle θ so that the range of the rocket is maximised, correct to the nearest 0.1 degree. [4]

4. **[10 marks]**

(MSPEC 2019:CA13)

The path of a particle is shown below. This particle moves so that its position vector $\underset{\sim}{r}(t)$ is given by $\underset{\sim}{r}(t) = \begin{pmatrix} -2\cos\left(\frac{t}{2}\right) \\ 1 - \sin(t) \end{pmatrix}$ metres, where t is the number of seconds the particle has been in motion.



(a) Determine the starting position of the particle and mark this as point A on the diagram above. [1]

b) Determine the initial velocity of the particle and illustrate this on the diagram above. [3]

4. (cont)

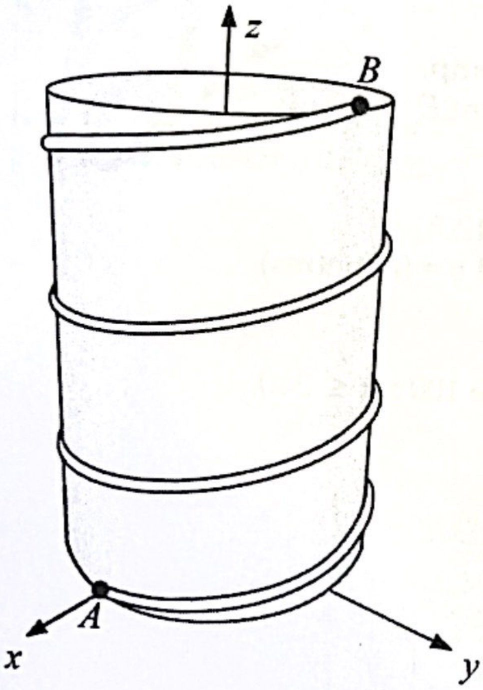
(c) Write the expression, in terms of trigonometric functions, for the distance the particle would travel in completing one circuit of the given path. Do **not** evaluate this expression. [3]

(d) Determine the Cartesian equation for the path of the particle. [3]

5. **[6 marks]**

(MSPEC 2020:CA12)

A cylindrical shaped tower has a path that spirals upwards from the ground to an observation deck at point B as shown in the diagram below. The path begins at point A on the ground and finishes at point B at the top.



Let t = time in seconds that a tourist has been walking along the spiral path. The tourist takes 65π seconds to reach point B.

The tourist's position on this path at any time t is given by:

$$\underset{\sim}{r}(t) = \begin{bmatrix} 10\cos(0.1t) \\ 10\sin(0.1t) \\ 0.2t \end{bmatrix} \text{ metres.}$$

(a) Determine the height of the observation deck above the ground, correct to the nearest 0.01 metres.

[1]

(b) Determine the tourist's velocity $\underset{\sim}{v}(t)$.

[2]

(c) Show that the tourist walks at a constant speed and determine this speed, correct to 0.01 metres per second.

[3]

(MSPEC 2021:CA19)

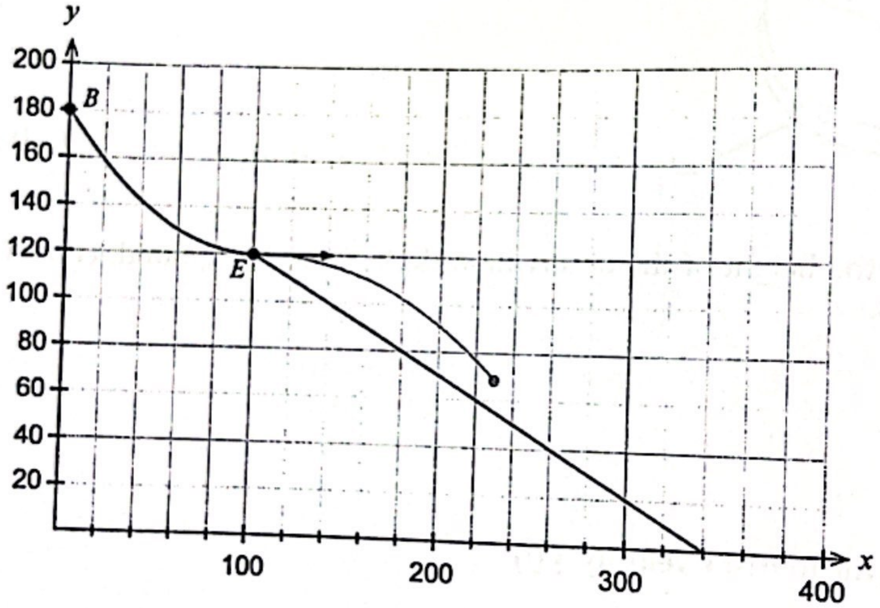
6. **[14 marks]**

Using the correct technique, Olympic ski jumpers can slow down their descent, by creating lift to counteract gravity. These jumpers must land successfully to have their distance recorded and land on sloped ground to prevent serious injury.

A skier begins his descent at point B accelerating down the ramp. At the end of the ramp the skier is travelling horizontally at point E at 32 metres per second (115.2 kilometres per hour).

Let t = the number of seconds in flight after point E (100, 120).
$h(t)$ = the height of the skier above the horizontal ground $y = 0$ (metres)
$x(t)$ = the horizontal position of the skier (metres)

The sloped ground for landing is given by $y = 170 - 0.5x$ where $100 \leq x \leq 340$.



The ski jumper's suit and skis decrease the horizontal velocity $x'(t)$ so that $x'(t) = 32e^{-0.05t}$.

(a) Show that $x(t) = 740 - 640e^{-0.05t}$.

[2]

6. (cont)

It is found that the expression for the position vector for the skier during the flight is given by:

$$\underset{\sim}{r}(t) = \begin{pmatrix} 740 - 640e^{-0.05t} \\ 120 - 2.5t^2 \end{pmatrix}$$

(b) Calculate the height of the skier above the sloped ground after 3 seconds of flight, correct to the nearest 0.01 metre. [3]

(c) Determine the vertical lift s (m/s^2) provided by the skier's suit and equipment in the descent if $\frac{d^2h}{dt^2} = s - 9.8$, where s is a constant. [3]

6. (cont)

It can be shown that the Cartesian equation for the skier's flight is given by:

$y = 120 - 1000\left(\ln\left(\frac{740 - x}{640}\right)\right)^2$ Note: The $\ln(f(x))$ function is learnt later in Unit 4. It is on the Classpad and is called the natural logarithm – $\ln x$ and e^x are inverse functions.

(d) Calculate the time taken for the skier to land on the sloped ground, correct to the nearest 0.01 second.

[3]

(e) Calculate the angle at which the skier impacts the sloped ground, correct to the nearest 0.1 degree.

[3]

1. **[6 marks]** *(MSPEC 2016S:CA8)*

The system of linear equations given below can be reduced in three stages to a form where it can be solved easily.

$$
\begin{aligned}
x + y + z &= 4 \quad ...R_1 \\
2x + 3y + z &= 8 \quad ...R_2 \\
3x + (3 - p)\,y + 2z &= 13 - p^2 \quad ...R_3
\end{aligned}
$$

(a) Two of the stages are given below.
In the space provided at the side of each stage, write the operation(s) that have been performed in terms of the rows of the previous system. [2]

$$
\begin{aligned}
x + y + z &= 4 \qquad ...R_1 \\
2x + 3y + z &= 8 \qquad ...R_2 \\
py + z &= p^2 - 1 \quad ...R_3
\end{aligned}
$$

$$
\begin{aligned}
x + y + z &= 4 \qquad ...R_1 \\
y - z &= 0 \qquad ...R_2 \\
py + z &= p^2 - 1 \quad ...R_3
\end{aligned}
$$

(b) Perform one further row operations so that the coefficient of z in R_3 is 0. [1]

(c) For each of $p = 1$ and $p = -1$ indicate how many solutions there are to the system of equations. If there is a unique solution, give that solution. If there is an infinite number of solutions, give the resulting solution when $z = -1$. [3]

(*MSPEC* 2016:CF06)

2. **[6 marks]**

(a) Solve the system of equations. [3]

$$\begin{aligned}x + y + z &= 4\\ 3x - y + z &= 8\\ 2x - y + z &= 0\end{aligned}$$

Suppose that the third equation in part (a) is changed to $2x - y + kz = 0$. The first two equations remain unchanged.

(b) Determine the value of the constant k so that the changed system of equations has no solution. [3]

3. **[6 marks]**

(MSPEC 2018:CF2)

(a) Solve the following system of equations: [3]

$4x - y - 2z = 5$
$2x + y - z = 4$
$x - y - z = 3$

3. (cont)

Consider another set of equations where k is a constant.

$$\begin{aligned} 2x - y - z &= 0 \\ x - 2y - z &= 2 \\ x - 2y + kz &= 6 \end{aligned}$$

It can be shown that this system of equations can be reduced to the following:

$$x = \frac{-2(k-1)}{3(k+1)}$$

$$y = \frac{-4(k+2)}{3(k+1)}$$

$$z = \frac{4}{k+1}$$

(b) Explain whether this system of equations will have a unique solution for all real values of k. If not, explain the geometric interpretation of this. [3]

4. **[7 marks]**

(MSPEC 2020:CF4)

Consider the equations for three planes, each written in Cartesian form:

$\Pi_1 \quad x + y + z = 4$

$\Pi_2 \quad x - y - z = 7$

$\Pi_3 \quad y + z = 1$

(a) Explain whether or not any of these planes are parallel. [2]

(b) Solve the given system of simultaneous equations. [3]

(c) Give the geometric interpretation of the solution for this system of equations. [2]

(MSPEC 2021:CA14)

5. [10 marks]

On a Saturday afternoon, three separate family groups visit their local cinema to watch a feature movie. The cinema names this as DollarDay where the ticket prices for adults, children and pensioners are charged in whole dollar amounts.

The table below indicates the number of people in each category and the total paid for each family group.

Group	Adults	Children	Pensioners	Total cost
1	2	4	–	\$108
2	3	6	–	\$162
3	2	5	2	\$152

Let a = the price for each adult (\$)
c = the price for each child (\$)
p = the price for each pensioner (\$)

(a) Formulate the equations that can be used to determine the ticket prices. [1]

(b) Using the equations formed, determine the total cost for a group consisting of 1 child accompanied by 2 pensioners. [2]

(c) Solve simultaneously the equations formulated in part (a). [2]

5. (cont)

(d) Explain the geometric interpretation of the equations and the simultaneous solution. [2]

Now assume that the price for an adult is greater than the price for a child and that the price for a pensioner is the lowest priced ticket.

(e) Determine the ticket prices for adults, children and pensioners on DollarDay. [3]

Chapter

11

Implicit Differentiation

1. **[5 marks]** *(3CDMAS 2014:CF5)*

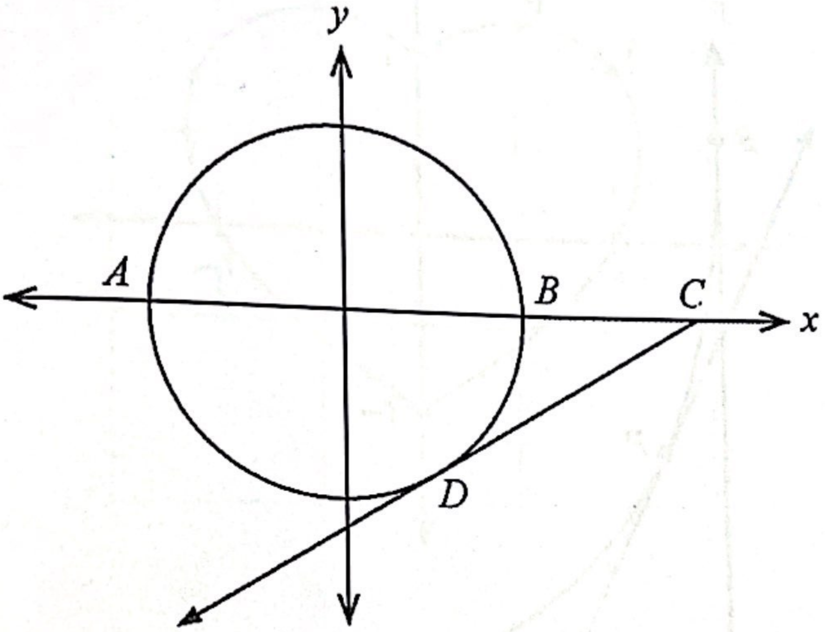
The tangent to the curve $y^2 = x^3$ at the point $P(1, 1)$ meets the x-axis at Q and the y-axis at R.

Determine the ratio $PQ : QR$.

2. **[3 marks]**

(MSPEC 2016:CA17a)

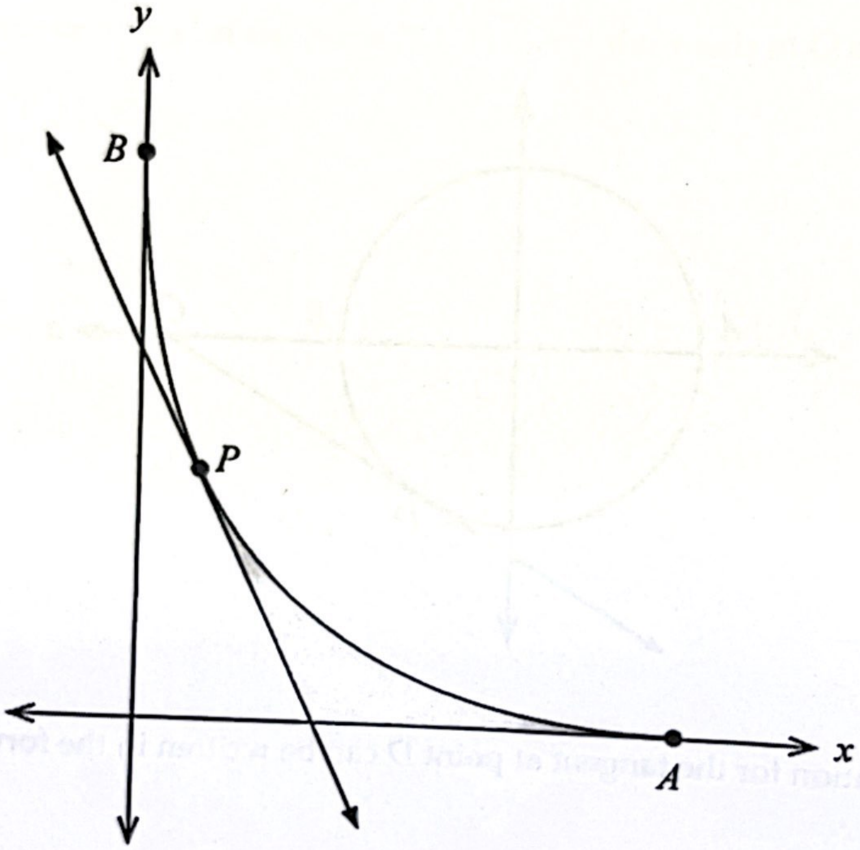
The diagram shows a circle with equation $x^2 + y^2 = 16$ with points A, B being the horizontal intercepts of this circle. DC is the tangent to the circle at point D, intersecting the x axis at point C. Point D has coordinates $(2, -2\sqrt{3})$.



Show that the equation for the tangent at point D can be written in the form $\sqrt{3}y = x - 8$.

3. **[3 marks]** *(MSPEC 2017:CA12a)*

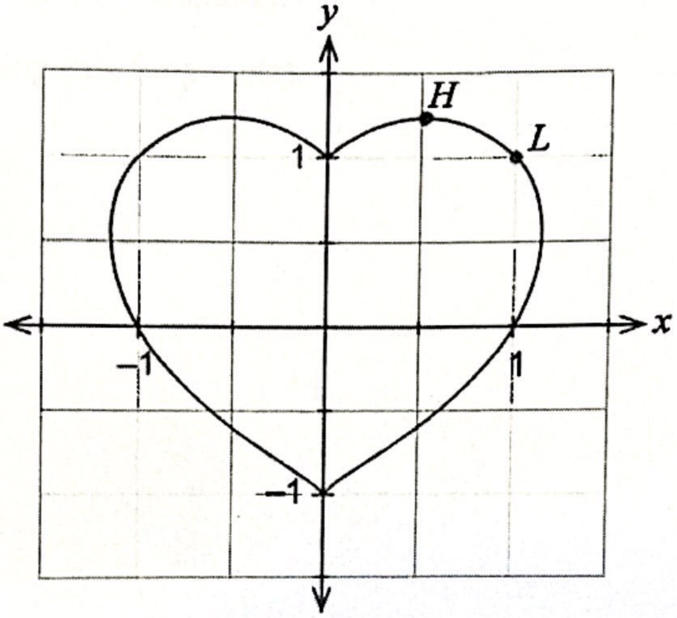
The diagram shows the curve with equation $\sqrt{x} + \sqrt{y} = 3$ where points A, B are the intercepts of this curve. A tangent is drawn to the curve at point $P(1,4)$.



Show that the equation of the tangent is $2x + y = 6$.

4. **[8 marks]** *(MSPEC 2018:CA20)*

The graph of $(x^2 + y^2 - 1)^3 = x^2y^3$ is shown below.



(a) By implicitly differentiating the given equation, obtain an equation relating x, y and $\frac{dy}{dx}$ [3]

(Note: Do **not** attempt to obtain $\frac{dy}{dx}$ as the subject of this equation.)

4. (cont)

(b) Determine the exact slope of the tangent to the curve at the point $L\ (1, 1)$.

At point H on the graph the curve is horizontal.

(c) Determine the coordinates of point H, correct to 0.001.

5. **[6 marks]**

(*MSPEC 2020:CF6a,b*)

Consider $f(x) = 2\tan(x)$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

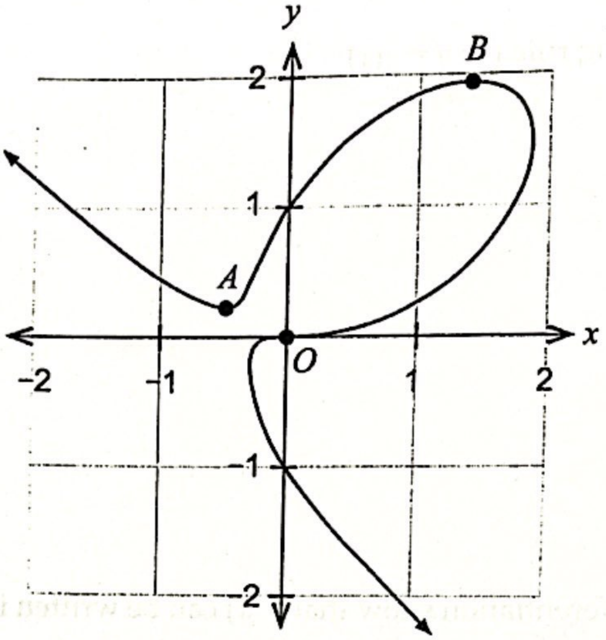
Let $g(x) = f^{-1}(x)$ be the inverse of function f.

(a) Determine the defining rule for $y = g(x)$. [2]

(b) By using implicit differentiation show that $g'(x)$ can be written in the form $\frac{a}{x^2 + b}$. [4]

6. **[6 marks]** *(MSPEC 2021:CA18)*

The equation $x^3 + y^3 = 3xy + y$ implicitly defines the curve shown below.



(a) Using implicit differentiation obtain the expression for $\frac{dy}{dx}$. [3]

6. (cont)

The slope of the curve at the origin O and points A and B is equal to zero.

(b) Show that the equation that determines the x coordinates for points A and B is given by $x^4 - 2x - 1 = 0$ and hence determine the coordinates for point A correct to 0.001. [3]

Chapter

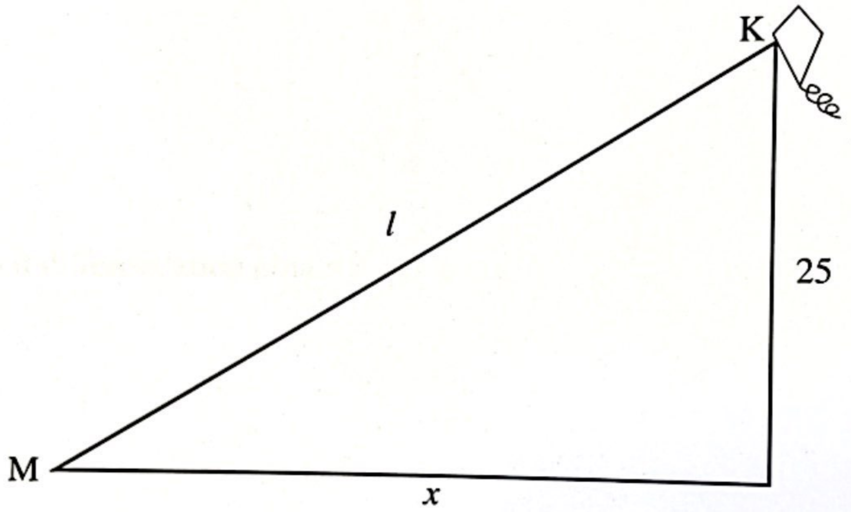
12

Related Rates

1. **[7 marks]** *(CA 2008:14)*

Melissa is flying a kite. She is standing still and the kite is moving away from her at a constant height of 25 m in a vertical plane that contains Melissa (M) and the kite (K). She keeps the string attached to the kite taut at all times, i.e. it forms the straight line segment MK, as shown in the diagram below. At a certain moment the length of the string is 65 metres and is increasing at the rate of 1.2 metres per second.

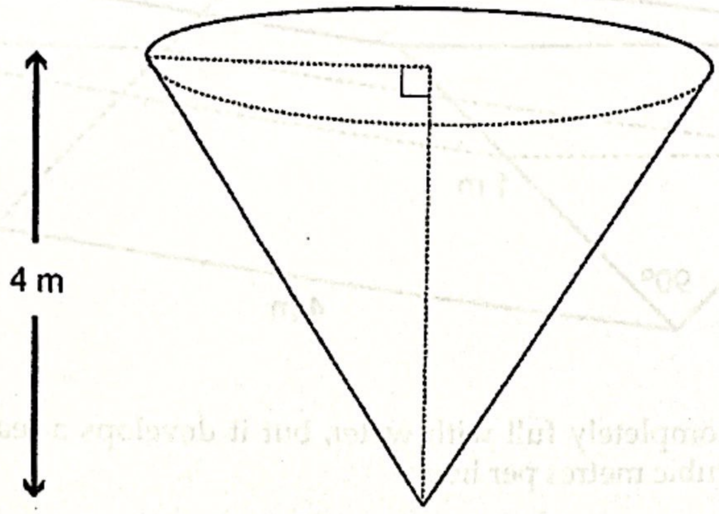
How fast is the kite moving at that moment?



2. **[5 marks]**

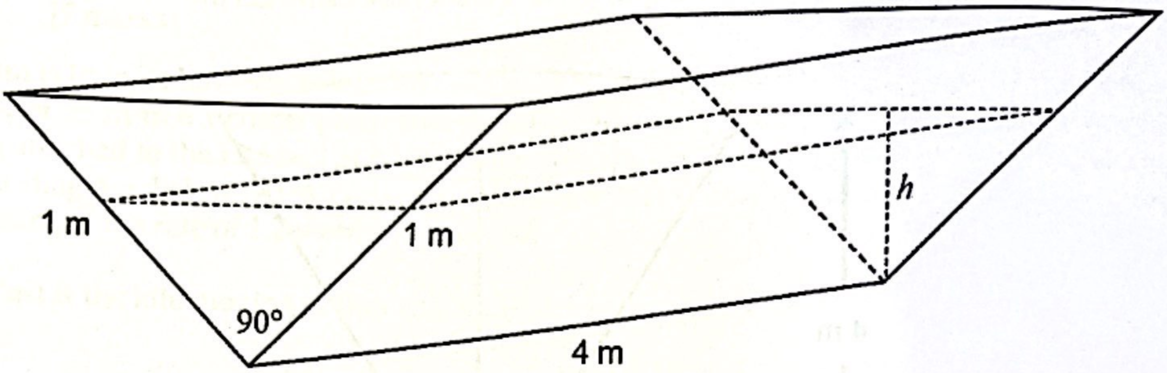
(MSPEC 2016S:CA11a)

The ethanol produced by a chemical factory is poured into a 4 m high conical container, with an upper diameter of 4 m, at a constant rate of 3 m^3 per minute. At what rate is the ethanol level rising in the container when the depth of the ethanol is exactly 2.5 m?



3. **[10 marks]** *(MSPEC 2016:CA15)*

A four metre long water tank, open at the top, is in the shape of a triangular prism. The triangular face is a right isosceles triangle with congruent sides of one metre length.



Initially the tank is completely full with water, but it develops a leak and loses water at a constant rate of 0.08 cubic metres per hour.

Let h = the depth of water, in metres, in the tank after t hours.

(a) Show that the volume of water in the tank V cubic metres, is given by the expression $V(h) = 4h^2$.

[2]

(b) Determine the rate of change of the depth, correct to the nearest 0.01 metres per hour, when the depth is 0.6 metres.

[3]

3. (cont)

Assume that the rate of leakage stays constant at 0.08 cubic metres per hour.

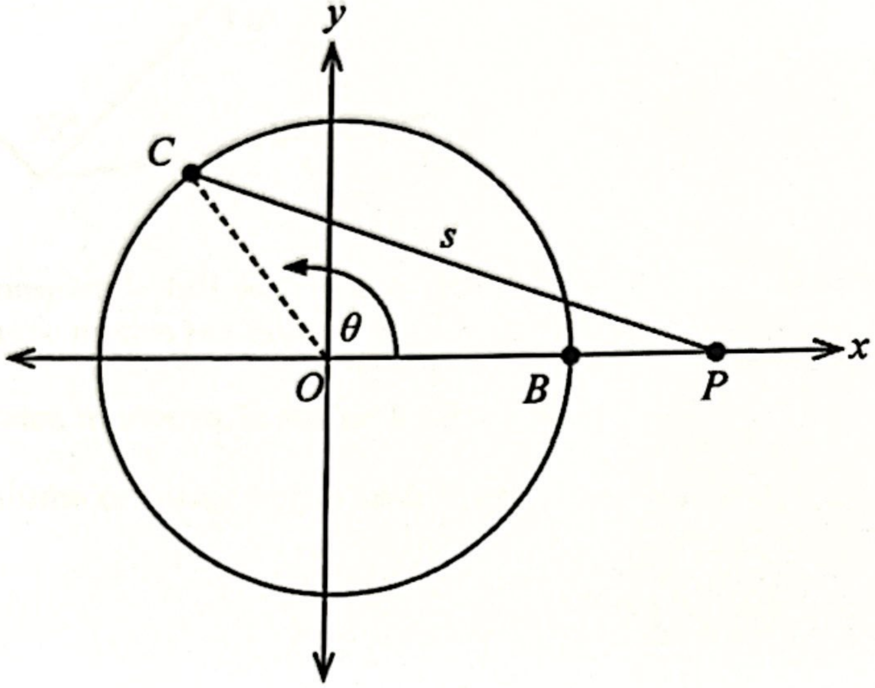
(c) Show that the differential equation that relates $\frac{dh}{dt}$ with the depth h is given by $\frac{dh}{dt} = -\frac{1}{100h}$. **[1]**

(d) Hence determine the relationship for the depth h at any time t hours. **[4]**

4. **[10 marks]**

A young child rides on a merry-go-round at a carnival. The merry-go-round has a radius of 5 metres and completes one revolution every 12 seconds. The parent of the young child stands and watches at point P, exactly 3 metres away from point B.

The ride begins at point B, when the child is closest to the parent, and the merry-go-round rotates in an anti-clockwise direction at a constant speed. At any point in time, point C is the position of the child on the merry-go-round.



Let t = the number of seconds the ride has been in progress (from starting at point B)
$s = PC$ = the distance that the child is from the parent (metres)
θ = size of $\angle BOC$ (radians)

(a) Show that $\frac{d\theta}{dt} = \frac{\pi}{6}$ radians per second.

[1]

(b) Show that $s^2 = 89 - 80\cos\theta$.

[1]

4. (cont)

(c) By differentiating $s^2 = 89 - 80\cos\theta$ implicitly with respect to time t, determine, correct to the nearest 0.01 metre per second, the rate at which the child is moving away from the parent when the ride has been in progress for 4 seconds.

[4]

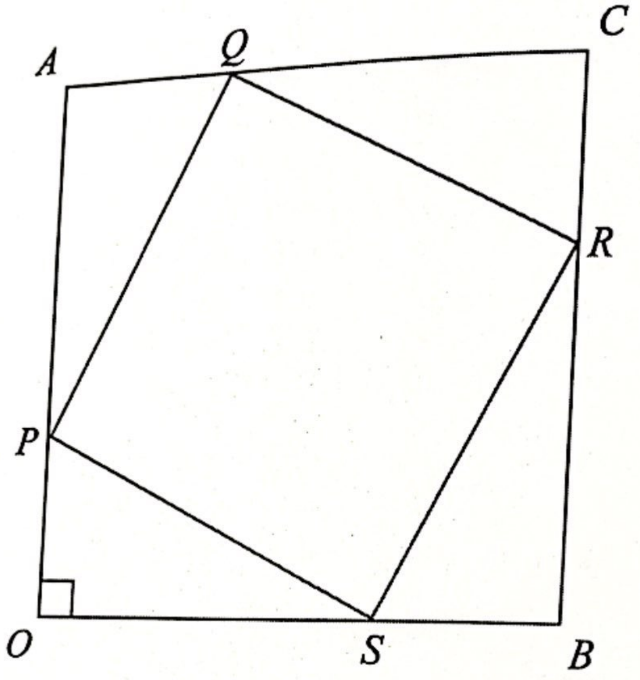
The parent notices that the child appears to move away from point P at varying speeds.

(d) Determine the value for $\cos\theta$ when the rate $\frac{ds}{dt}$ is a maximum. [4]

(MSPEC 2020:CA20b)

5. **[4 marks]**

Consider square $OACB$ where point O is the origin. It is known that $OA = 10$ cm and that point P is moving away from the origin at a speed of 0.2 cm per second. This means that points Q, R and S are moving at the same speeds along their respective sides.



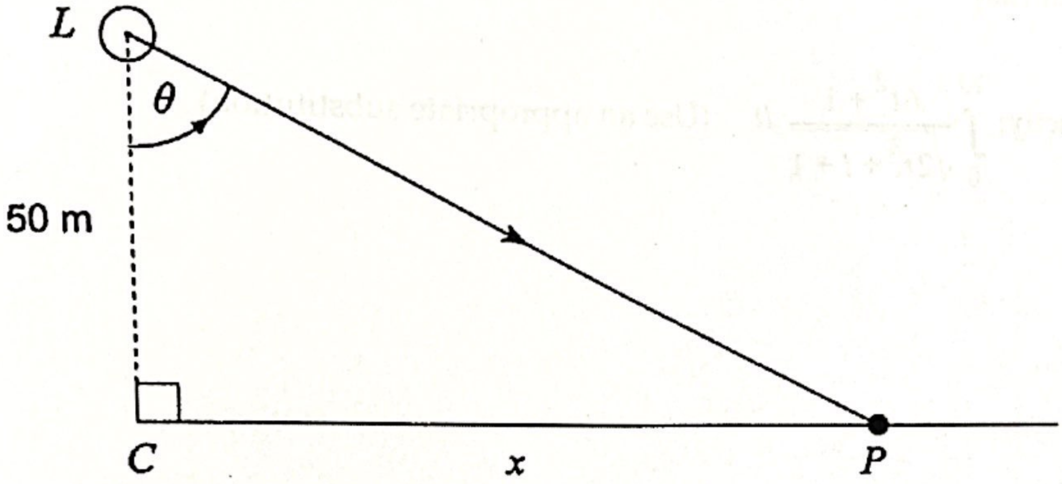
Let x = the distance OP.

Determine the rate at which the area of square $PQRS$ is changing when $x = 3$ cm.

6. **[5 marks]**

(MSPEC 2021:CA09)

A beam of light completes three revolutions each minute from a lighthouse L that is 50 metres from a coastline. Determine the speed of the beam of light moving along the coast when it is at point P, 100 metres up the coast, correct to the nearest 0.01 metres per second.



Chapter

13

Integration Techniques

(3CDMAS 2011:CF6)

1. **[5 marks]**

Evaluate exactly: $\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt$ (Use an appropriate substitution)

2. **[6 marks]**

(MSPEC 2016:CF04)

(a) Express $\dfrac{x-8}{(x+2)(x-3)}$ in the form $\dfrac{a}{x+2}+\dfrac{b}{x-3}$. [3]

(b) Hence determine $\displaystyle\int\frac{x-8}{(x+2)(x-3)}\,dx$. [3]

3. **[5 marks]** *(MSPEC 2016:CA09)*

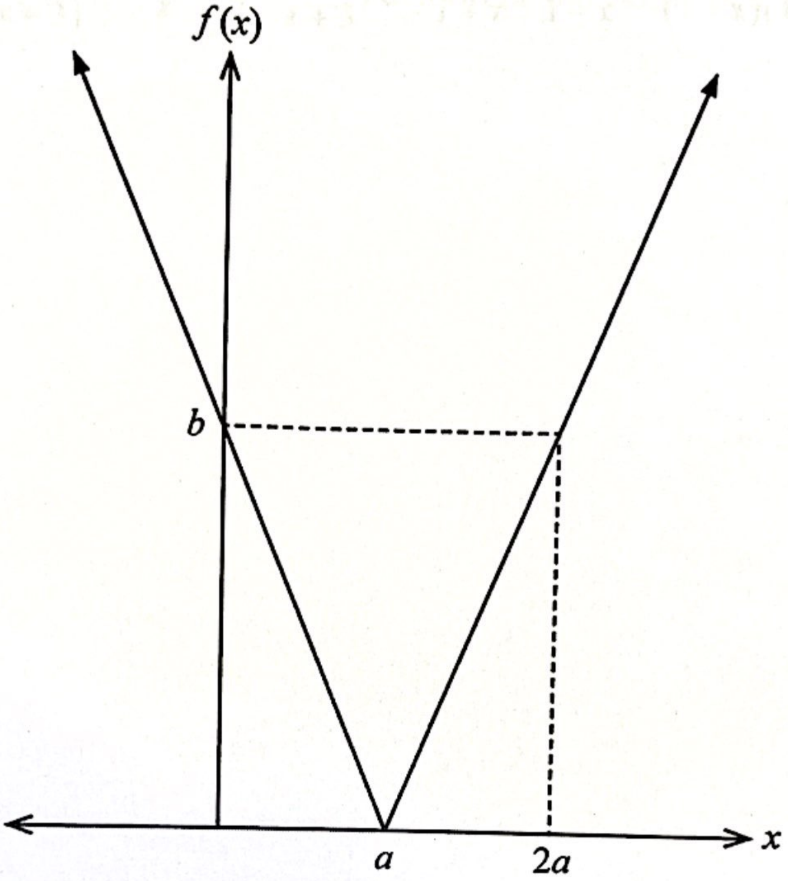
Consider the integral $I = \int x\sqrt{(1+x)^n}\ dx$, where n is any positive integer.

Using the substitution $u = 1 + x$ and an appropriate anti-derivative, develop a simplified expression for I in terms of x and n.

4. **[5 marks]**

(MSPEC 2017:CA16b)

Function f is defined by its graph shown below. The constants $a,\ b > 0$ where $b > a$.



Given $f(x) = \frac{b}{a}\,|x - a|$

By using the substitution $u = 2x - a$, determine an expression, in terms of $a,\ b$, for the value of $\int_{\frac{a}{2}}^{a} f(2x - a)dx$.

(MSPEC 2018:CF6)

5. **[5 marks]**

(a) Given that $\frac{2}{(x+1)(x-1)} = \frac{a}{x-1} + \frac{b}{x+1}$ determine the values for a and b. [2]

(b) Hence determine $\int \frac{1}{x^2 - 1}\, dx$. [3]

6. **[5 marks]**

(MSPEC 2019:CF3)

(a) Given that $\frac{2x^2 + 5x + 6}{x^2(x+3)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x+3}$, determine the values of *a*, *b* and *c*. [2]

(b) Hence determine $\int \frac{2x^2 + 5x + 6}{x^2(x+3)}\, dx$. [3]

(MSPEC 2020:CF6c,d)

7. **[7 marks]**

(a) Show that $\frac{3x^2+2x+6}{(x^2+4)(x-3)}$ can be expressed as $\frac{q}{x^2+4}+\frac{r}{x-3}$ and hence determine the values for q and r. [3]

(b) Hence determine $\int \frac{3x^2+2x+6}{(x^2+4)(x-3)}\,dx$. [4]

Hint: $\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right)=\frac{a}{x^2+a^2}$, a is a constant.

8. **[5 marks]** *(MSPEC 2020:CF7)*

Evaluate $\int_{-1}^{7} \frac{3x}{\sqrt{x+2}}\,dx$ exactly using the substitution $u = \sqrt{x+2}$.

(MSPEC 2021:CF03)

9. [5 marks]

Using an appropriate substitution, determine the exact value for $\int_{2}^{3} 15x\sqrt{x-2}\,dx$.

[5 marks]

(MSPEC 2021:C

Given that $\frac{7x^2 - 12x + 2}{(x-2)(x^2+2)} = \frac{a}{x-2} + \frac{bx}{x^2+2}$ determine the values of *a* and *b*.

Hence determine $\int \frac{7x^2 - 12x + 2}{(x-2)(x^2+2)}\, dx$.

[3]

Chapter

14

Integration Techniques and Trigonometric Functions

(MSPEC 2016:C

1. **[7 marks]**

Evaluate the following definite integrals exactly.

(a) $\displaystyle\int_{0}^{\frac{\pi}{4}} 12\ \sin^4 2x \cos 2x\, dx$ Put $u = \sin 2x$

(b) $\displaystyle\int_{0}^{\frac{1}{2}} \tan^2\left(\frac{\pi x}{2}\right) dx$

2. **[7 marks]** *(MSPEC 2017:CF3)*

Consider the definite integral $\int_0^1 \frac{x^2}{(1+x^2)^2}\,dx$.

(a) By using the substitution $x = \tan u$, show that $\int_0^1 \frac{x^2}{(1+x^2)^2}\,dx = \int_a^b \sin^2 u \;\; du$ and state the values of a, b. [4]

(b) Hence evaluate $\int_0^1 \frac{x^2}{(1+x^2)^2}\,dx$ exactly. [3]

(MSPEC 2018:CF5)

3. **[4 marks]**

Using the substitution $u = \cos(2x)$, evaluate exactly the definite integral

$$\int_0^{\frac{\pi}{4}} \cos^{1008}(2x)\sin(2x)\,dx.$$

4. **[7 marks]** *(MSPEC 2018:CF9)*

(a) By using an appropriate trigonometric identity, simplify in terms of u, the expression $x^2 - 2x + 4$ where $x = \sqrt{3}\ \tan(u) + 1$. [2]

(b) Hence evaluate $\displaystyle\int_1^2 \frac{dx}{(x^2 - 2x + 4)^{\frac{3}{2}}}$ exactly. [5]

5. **[4 marks]** *(MSPEC 2019:CF1)*

Using the identity $2\sin A\cos B = \sin(A+B) + \sin(A-B)$, evaluate exactly the definite integral

$$\int_0^{\frac{\pi}{2}} 6\sin\left(\frac{5x}{2}\right)\cos\left(\frac{x}{2}\right)dx.$$

6. **[6 marks]** (MSPEC 20

Using the substitution $x = 2\sin\theta$, evaluate exactly $\displaystyle\int_0^{\sqrt{3}} \sqrt{1 - \frac{x^2}{4}}\, dx$.

(MSPEC 2020:CF1

7. **[3 marks]**

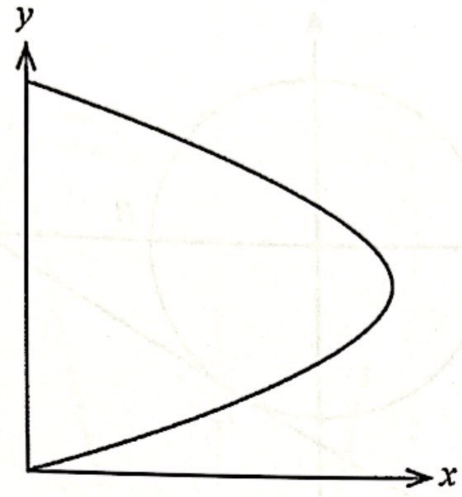
Evaluate exactly $\int_0^{\pi}(4\cos^2 x - \sin x)\ dx$.

15

(MSPEC 2016:CA13)

1. **[5 marks]**

The graph of the curve $2x = \sin(y)$ is sketched for $0 \le y \le \pi$.



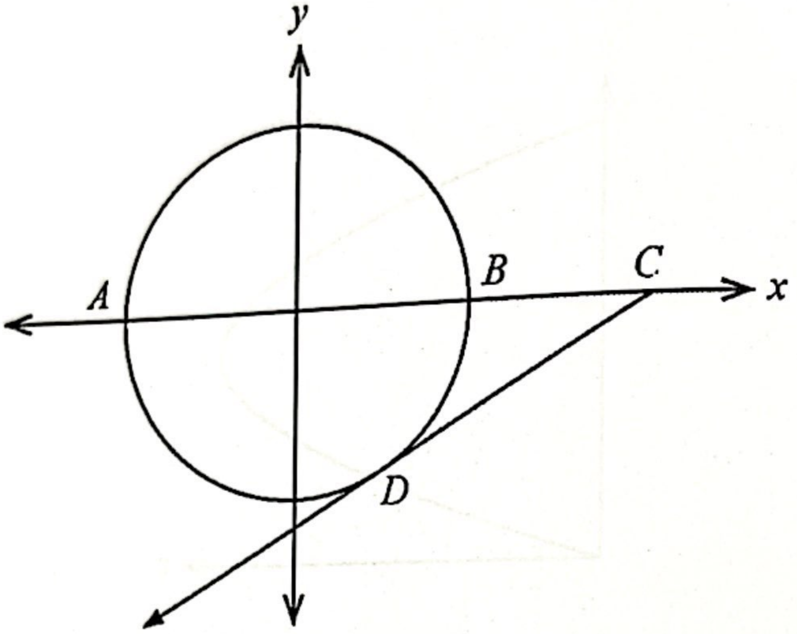
(a) Determine the expression for $\dfrac{dy}{dx}$ in terms of y. [2]

(b) Determine the area of the region bounded by the curve $2x = \sin(y)$ and the y axis. [3]

(MSPEC 2016:CA17b,c)

2. **[5 marks]**

The diagram shows a circle with equation $x^2 + y^2 = 16$ with points A, B being the horizontal intercepts of this circle. DC is the tangent to the circle at point D, intersecting the x axis at point C. Point D has coordinates $(2, -2\sqrt{3})$.



(a) Determine the coordinates of point C given the equation of the tangent at D is $\sqrt{3}y = x - 8$. [1]

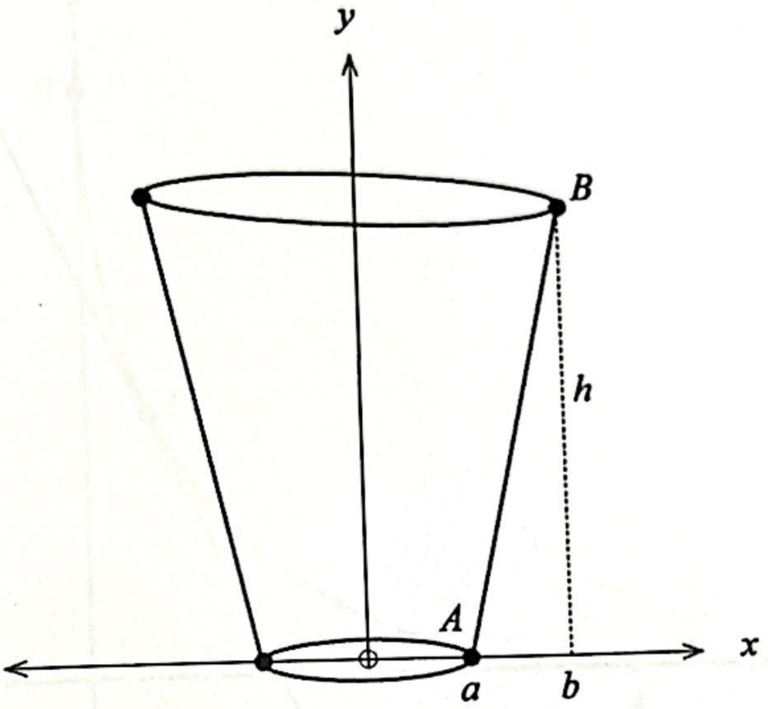
The region bounded by the arc AD, the line segment $\overline{DC}$ and the x axis is rotated about the x axis.

(b) Determine the volume of the resulting solid, correct to the nearest 0.01 cubic units. [4]

[5 marks]

(MSPEC 2017:CF8)

3. The inner surface of a drinking glass can be modelled by rotating the line segment $\overline{AB}$ about the y axis, as shown in the diagram below. The radius of the glass at the bottom is a cm and the radius at the top is b cm. The height of the glass is h cm.



The equation for $\overline{AB}$ is $y = \left(\frac{x-a}{b-a}\right) h$.

(a) Write an expression, in terms of a definite integral, for the volume of liquid contained by the glass when it is full. [2]

3. (cont)

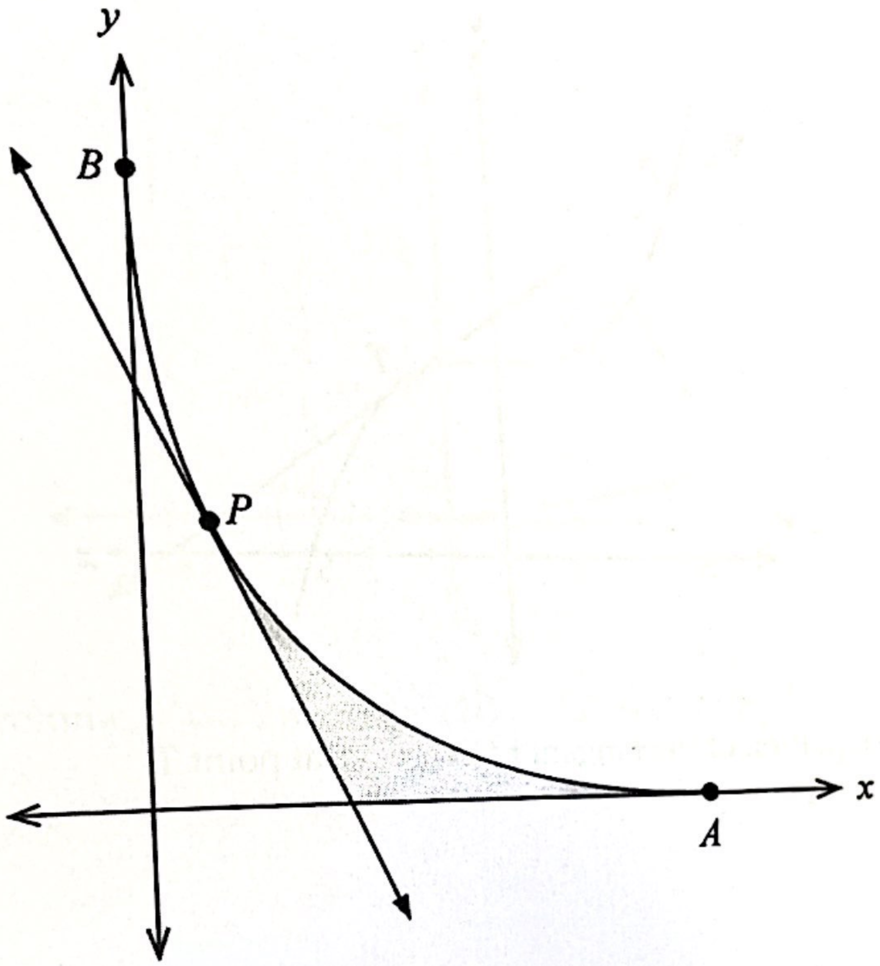
(b) By using an anti-derivative, obtain a simplified expression/formula (in terms of a, b and h) for the volume of liquid contained by the glass when it is full. [3]

[5 marks]

4\.

(MSPEC 2017:CA12b)

The diagram shows the curve with equation $\sqrt{x} + \sqrt{y} = 3$ where points A, B are the intercepts of this curve. A tangent is drawn to the curve at point $P(1,4)$.



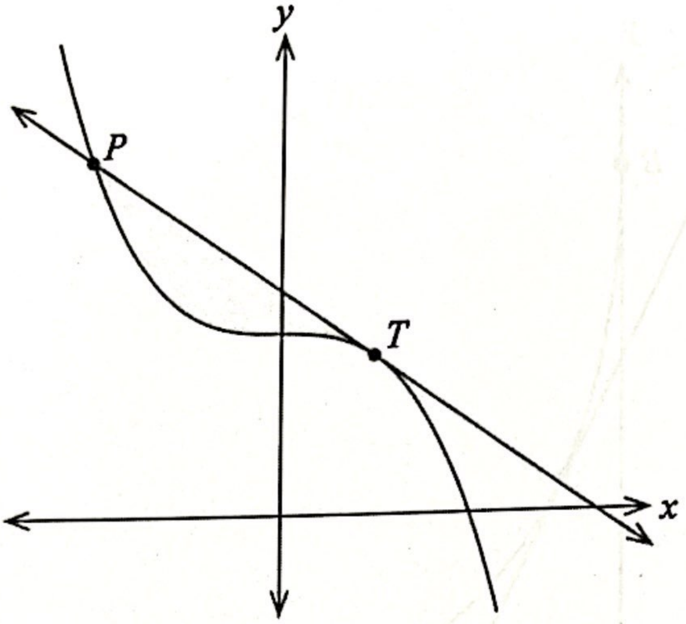
The shaded region on the diagram is bounded by the curve, the tangent whose equation is $2x + y = 6$, and the x axis.

Determine the exact area of the shaded region.

(MSPEC 2018:CA15)

5. [8 marks]

Part of the graph of $x^3 + 8y = 64$ is shown below. A tangent is drawn to the curve at point $T\ (2, 7)$, intersecting the curve again at point P.



(a) Determine the equation of the tangent to the curve at point T. [2]

(b) Determine the area of the shaded region. [3]

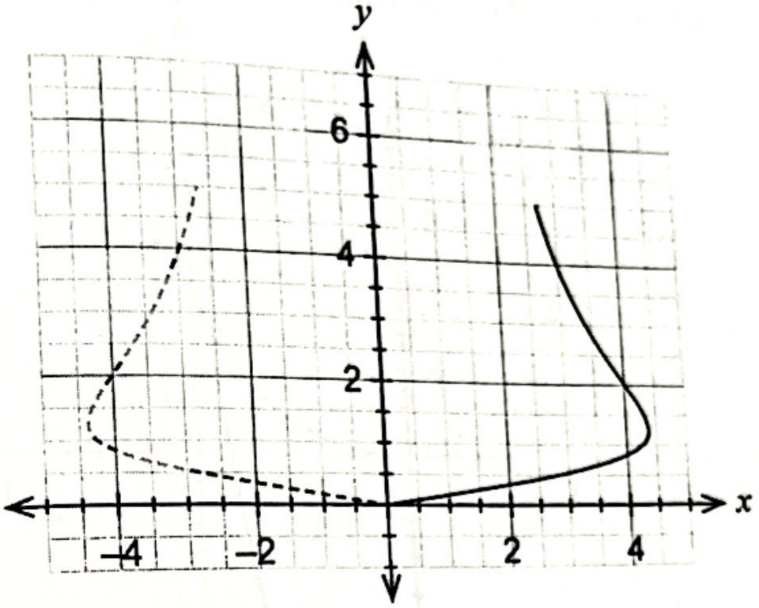
The shaded region is then rotated about the x axis.

(c) Calculate the volume of the resulting solid, correct to 0.01 cubic units. [3]

[5 marks]

6\. (MSPEC 2019:CF8)

The top part of a wine glass is modelled by rotating the graph of $x^2 = y^2(36 - x^2y)$ from $y = 0$ to $y = 5$ about the y axis as shown below. Dimensions are measured in centimetres.



(a) Show that the volume, V cm^3, when the glass is full is given by [1]

$$V = \pi \int_0^5 \frac{36y^2}{1 + y^3}\, dy.$$

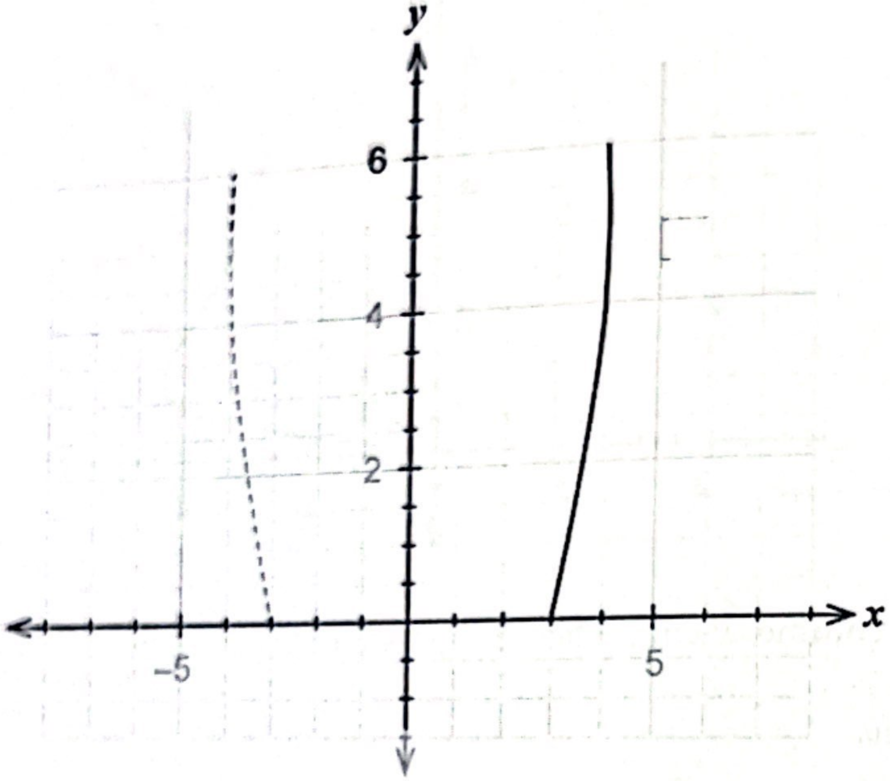
[4]

(b) Determine the exact volume V cm^3.

(MSPEC 2020:CA9)

7. [4 marks]

The shape of a small wine glass is modelled by revolving the curve $\sin\left(\frac{y}{\pi}\right) = x - 3$ about the y axis, where $0 \le y \le 6$. All dimensions are in centimetres.

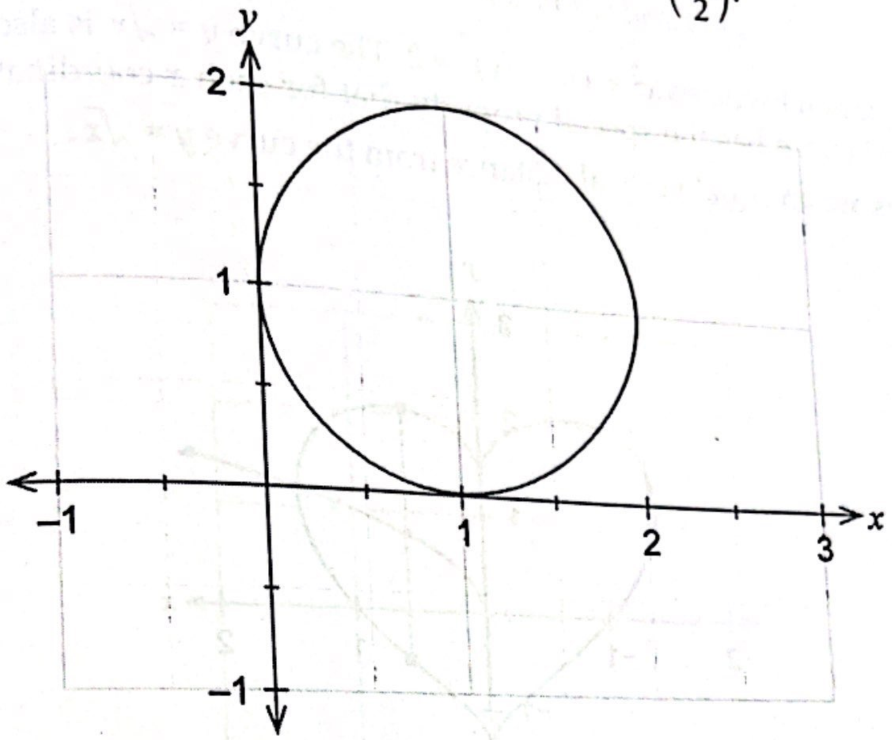


Calculate, correct to the nearest 0.01 cm, the depth of wine in the glass if it is to contain 80% of its maximum volume.

[5 marks]

(MSPEC 2020:CA16)

8\. The curve shown below is given by the equation $(y - 1)^2 = \sin\left(\frac{\pi x}{2}\right)$.



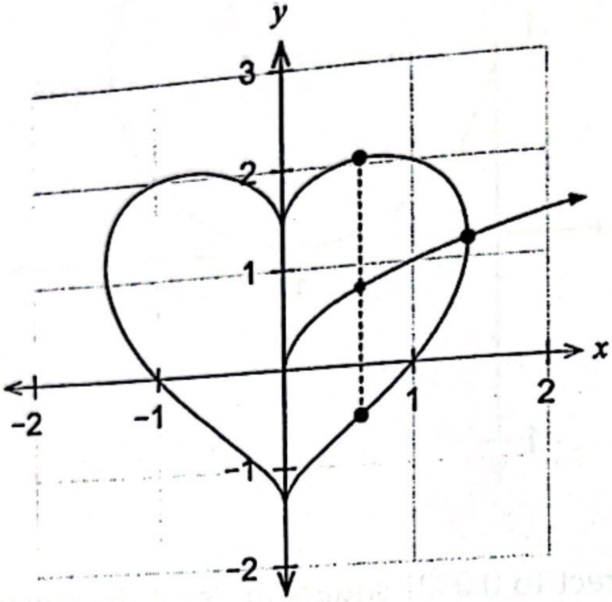
(a) Calculate the area, correct to 0.0001 square units, of the region that forms the interior of this curve. [3]

(b) By using the answer to part (a), determine whether this curve is a circle. Explain your answer. [2]

9. **[9 marks]**

The heart-shaped figure shown is given by the equation $x^2 + (y - \sqrt{|x|})^2 = 2$.

For $x \geq 0$, this equation becomes $x^2 + (y - \sqrt{x})^2 = 2$. The curve $y = \sqrt{x}$ is also drawn. This heart-shaped curve has the special property that for each x coordinate in its domain its two y coordinates are an equal vertical distance from the curve $y = \sqrt{x}$.



(a) Explain why the domain for the curve given by $x^2 + (y - \sqrt{x})^2 = 2$ is $0 \leq x \leq \sqrt{2}$. [2]

(b) Show that the total area enclosed by the heart-shaped figure is given by:

$$Area = 4\int_0^{\sqrt{2}} \sqrt{2 - x^2}\,dx.$$ [2]

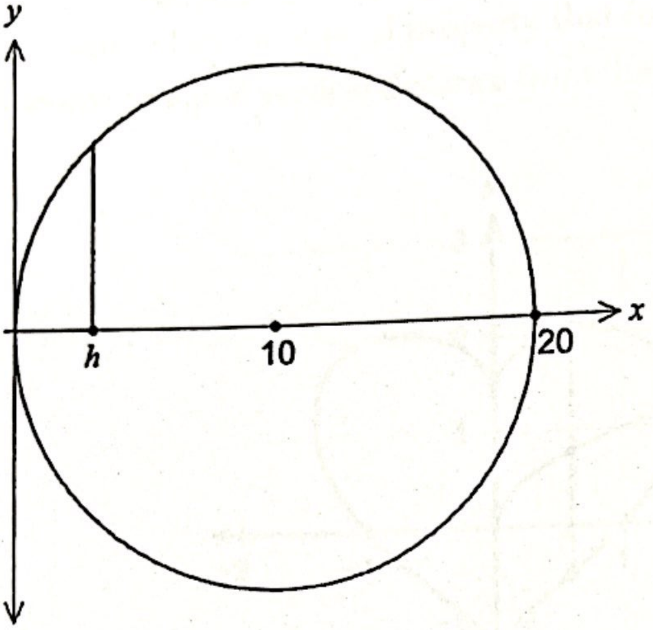
9. (cont)

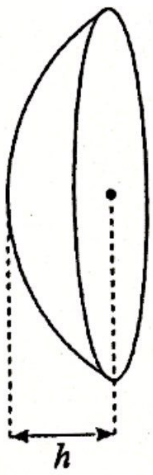
(c) By using the substitution $x = \sqrt{2}\sin\theta$, evaluate the total area enclosed by the heart-shaped figure, and hence see why it can be said that 'π is at the heart of mathematics'. [5]

(MSPEC 2021:CA13)

10. **[5 marks]**

A solid spherical cap with depth h is part of a solid sphere with radius 10 cm. This cap can be generated by revolving the region above the x-axis from 0 to h.





(a) Show that the equation for the circle shown above is $x^2 + y^2 = 20x$. [1]

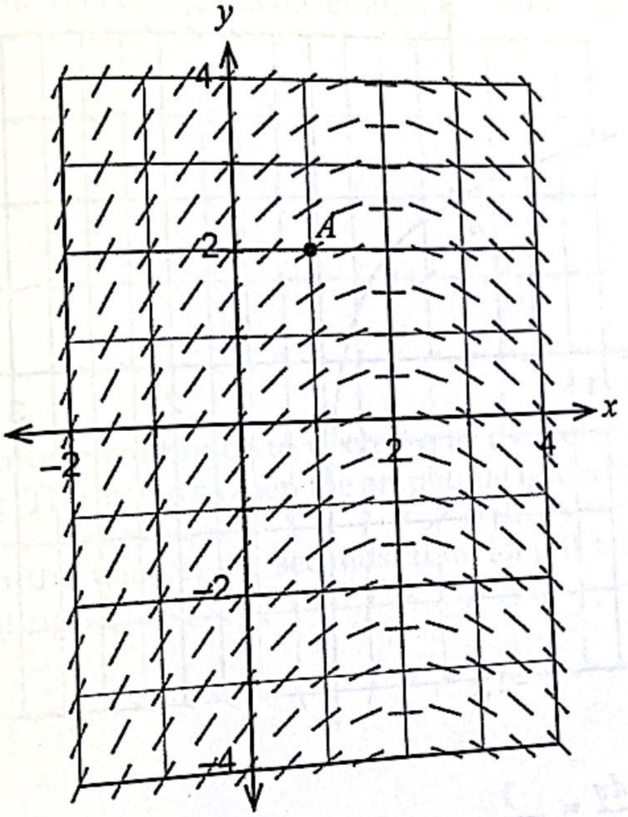
(b) Develop an expression for the volume of the spherical cap in terms of h. [4]

Differential Equations

Chapter 16

[7 marks] (MSPEC 2016:CA18)

1. A first-order differential equation has a slope field as shown in the diagram below.



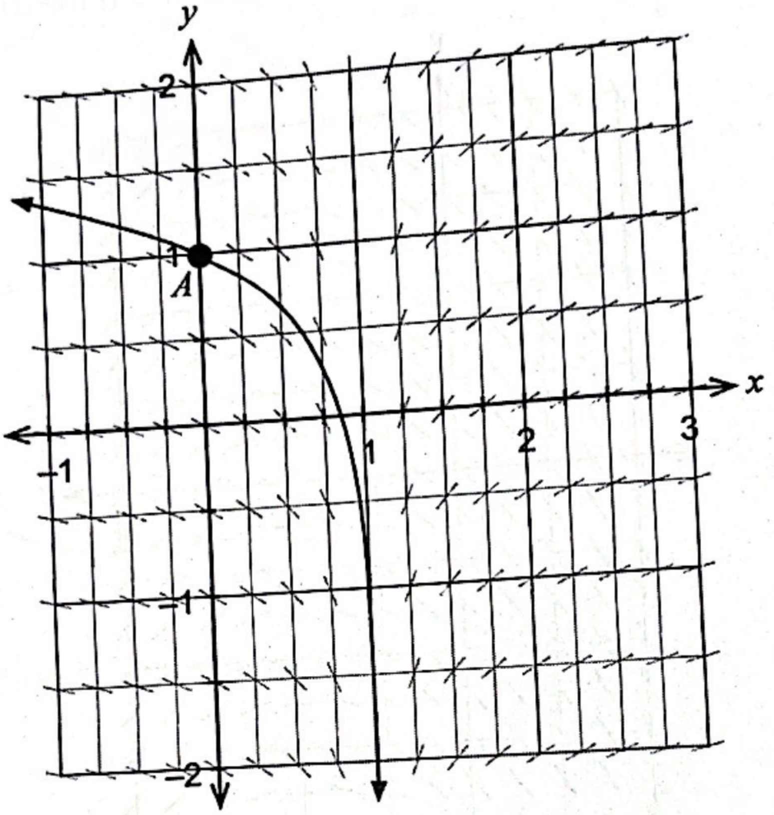
(a) Determine the general differential equation that would yield this slope field. [3]

(b) Determine the equation for the curve $y = f(x)$ containing point A given that the slope field at point $A(1,2)$ has a value of 0.5. [4]

(MSPEC 2017:CA11)

2. **[7 marks]**

A series of magnets is placed under a glass pane and some iron filings are sprinkled onto the glass. The orientation or slope of the iron filings, as determined by the magnetic field, is shown below. One of the lines of magnetic force that passes through the point $A\ (0,1)$ is also shown.



The slope field is given by $\dfrac{dy}{dx} = \dfrac{1}{2x-2},\ x \neq 1$.

(a) Determine the value of the slope field at the point $A\ (0,1)$. [2]

(b) Explain the orientation of the iron filings at $x = 1$. [1]

(c) Determine the equation for the line of force that passes through the point $A\ (0,1)$. [4

[5 marks]

(MSPEC 2017:CA17c,d)

3. After t seconds, a small mass attached to a spring, oscillates about a fixed point. Friction reduces the amplitude of the oscillation according to the equation

$$\frac{dA}{dt} = -0.4A \quad \text{Assume } A(0) = 8 \text{ cm}$$

(a) Determine the function $A(t)$ that gives the amplitude of the mass. [2]

As time passes, the amplitude continues to decrease to the point at which the small mass appears to stop oscillating. This occurs when the amplitude is less than 0.01 cm.

(b) Determine, correct to the nearest 0.1 seconds, how long it takes for the small mass to appear to stop oscillating. [3]

(MSPEC 2018:CA18)

4. **[13 marks]**

A rumour that the Federal Government plans to cut university funding begins to spread around a campus. There is a combined total of 1600 students and staff at this university.

One hundred people know of this rumour via a social media post at 8 am one morning.

Let $N(t)$ = the number of people at the university who have heard the rumour at t hours after 8 am. It is found that the rate at which the rumour spreads is given by the equation:

$$\frac{dN}{dt} = kN(1600 - N).$$

At 8 am the rumour was spreading at a rate of 60 people per hour.

(a) Show that $k = 0.0004$. [2]

(b) At 11 am there were approximately 500 people who had heard the rumour. Using the increments formula, determine the approximate number of people who learn of this rumour between 11 am and 11.15 am. [3]

(cont)

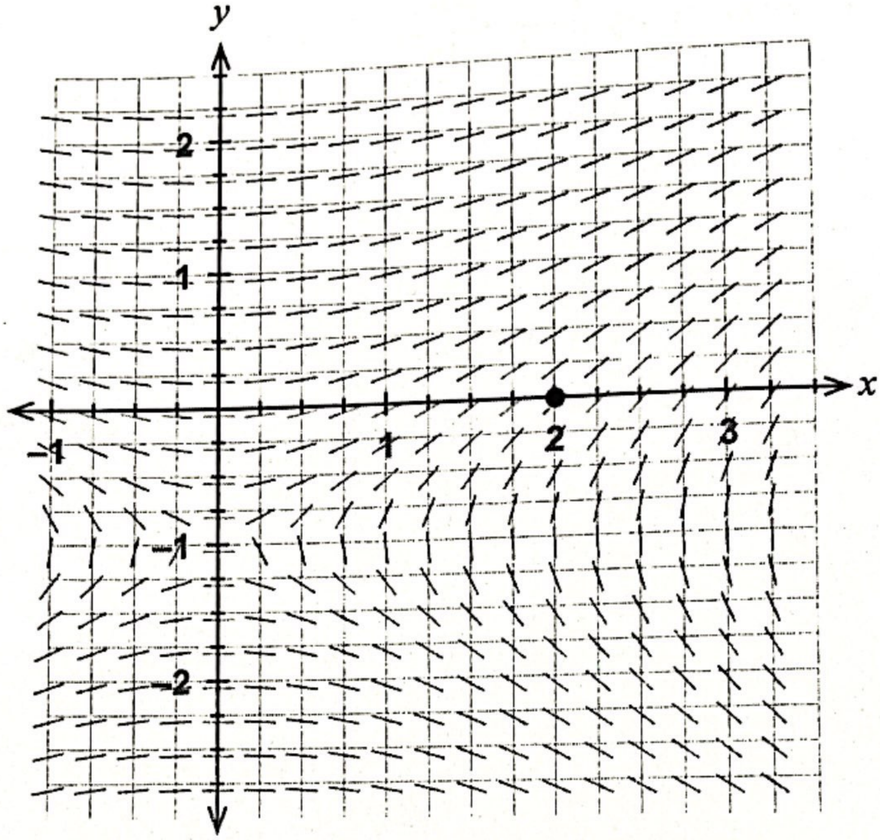
) Given that $\frac{1}{N(1600-N)} = \frac{1}{1600}\left(\frac{1}{N} + \frac{1}{1600-N}\right)$, use the separation of variables technique to show that $N(t)$ is given by $\frac{N}{1600-N} = \frac{e^{0.64t}}{15}$. [4]

) At what time, correct to the nearest minute, does the rumour spread at the fastest rate? [4]

(MSPEC 2019:CA11)

5. **[6 marks]**

The slope field given by $\frac{dy}{dx} = \frac{x}{2y+2}$ is shown in the diagram below.



(a) Calculate the value of the slope field at the point (2,0). [1]

(b) On the diagram above, draw the solution curve that contains the point (2,0). [2]

(c) Determine the equation for the solution curve that contains the point (2,0). [3]

6. **[8 marks]** *(MSPEC 2019:CA17)*

In Australia, the killing of humpback whales was banned in 1963.

At the end of 2018, 55 years later, the population P of migrating humpback whales off the coast of Western Australia was estimated at 30 000, i.e. $P(55) = 30\,000$.

(a) Assuming that the population of humpback whales had been increasing at an instantaneous rate equal to 10% of the population, estimate the number of humpback whales at the end of 1963. [3]

To model the growth in the population from the end of 2018, a marine biologist suggests that the rate of growth be modelled by the equation below.

$$\frac{dP}{dt} = 0.1P - \frac{P^2}{700\,000}$$

The biologist re-defines $P(0) = 30\,000$, i.e. t = number of years from the end of 2018.

(b) If $P(t)$ is written in the form $P(t) = \dfrac{a}{1 + be^{-ct}}$, determine the values of the constants a, b and c. [2]

6. (cont)

(c) Hence determine the year during which the population of humpback whales off the coast of Western Australia will reach double that estimated at the end of 2018. [2]

(d) State the major difference in the variation in the population $P(t)$ using the model in part (b) compared with that in part (a). [1]

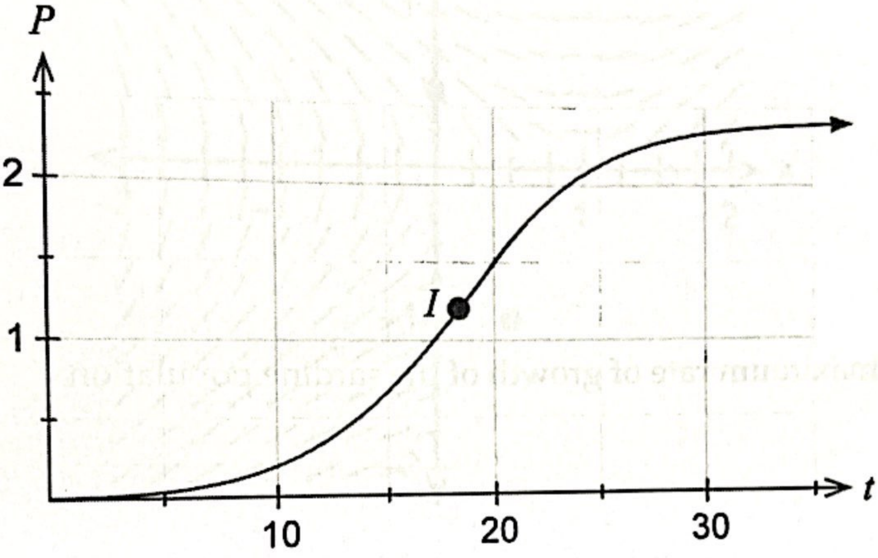
7. **[11 marks]**

(MSPEC 2020:CA19)

The population $P(t)$ of sardines in an ocean, measured in million tonnes after t years, was modelled by the logistic equation:

$$P(t) = \frac{2.4}{1 + 239e^{-0.3t}}$$

The graph of this model is shown below. This graph contains a point of inflection at point I.



(a) Calculate the size of the sardine ocean population at $t = 0$. [2]

(b) Rewrite the logistic equation in the form $\frac{dP}{dt} = r\,P(k - P)$, stating clearly the values for r and k. [2]

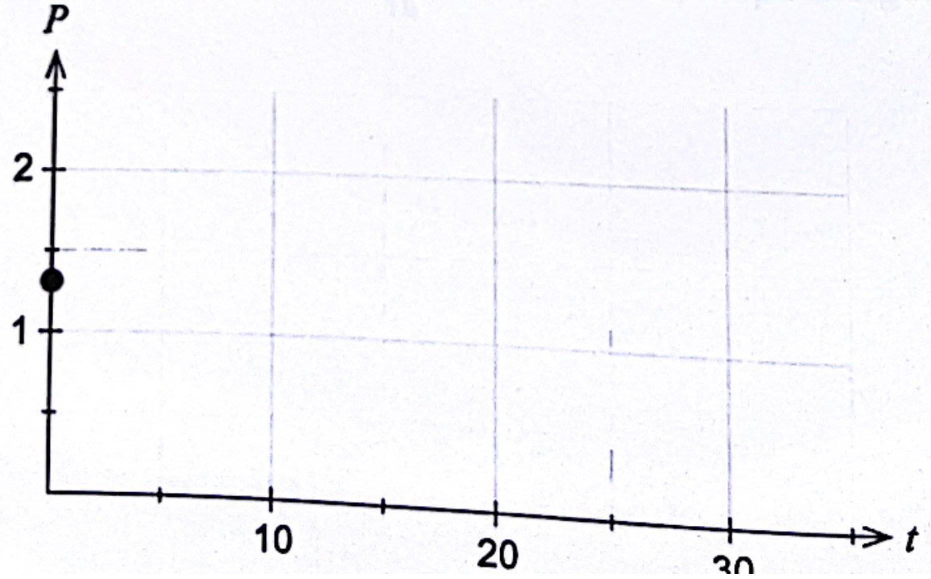
7. (cont)

(c) When the sardine population is 500 000 tonnes, use the technique of increments to calculate the approximate change in population in the next month. [3]

(d) Determine the maximum rate of growth of the sardine population. [2]

Suppose that the initial population of sardines was 1.3 million tonnes.

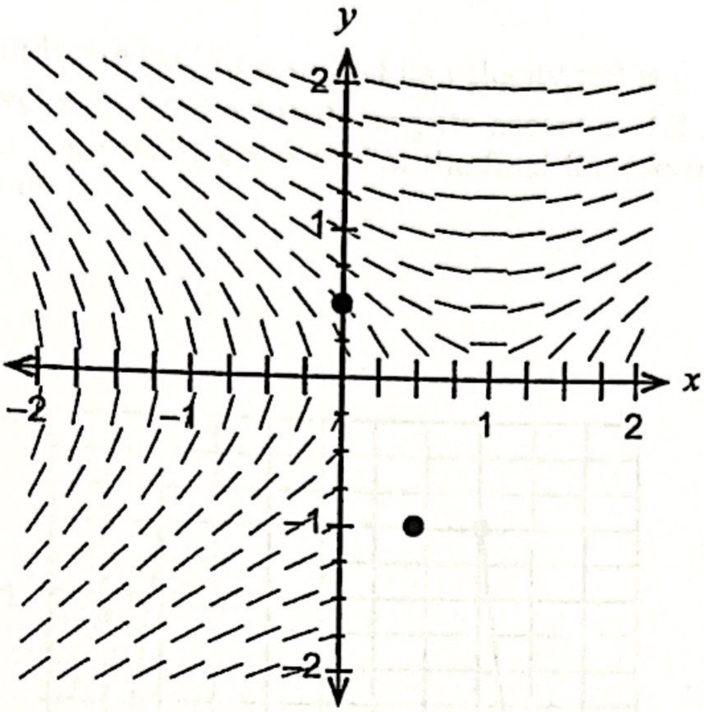
(e) Assuming that the rate of growth is still given by $\frac{dP}{dt} = r\,P(k - P)$, sketch the graph of the population growth on the axes below. Explain your graph. [2]



8. **[8 marks]**

(MSPEC 2021:CA10)

Part of the slope field given by $\frac{dy}{dx} = \frac{x-1}{2y}$ is shown below.



(a) Calculate and draw the slope field at the point (0.5, –1). [3]

8. (cont)

(b) Determine the equation of the solution curve that contains the point (0, 0.5). [3]

(c) Draw the solution curve that contains the point (0, 0.5). [2]

(MSPEC 2017:CA17a,b)

2. **[6 marks]**

After t seconds, the displacement x centimetres of a small mass attached to a spring, oscillates about a fixed point O according to the differential equation $\frac{d^2x}{dt^2} = -\pi^2 x$.

The initial velocity is 8π centimetres per second and the initial displacement is zero.

(a) Determine the function $x(t)$ that gives the displacement of the mass at time t. [3]

(b) Calculate the distance the mass travels during the first 5 seconds. [3]

3. **[5 marks]**

(MSPEC 2018:CA13)

A particle travels in a straight line so that its velocity v cm/sec and displacement x cm are related by the equation:

$$v = \frac{2}{x}$$

(a) Determine the acceleration a in terms of its displacement x. [2]

It is known that the initial displacement of the particle is $x = 2$ cm.

(b) Determine the displacement x as a function of time t. [3]

(MSPEC 2019:CA18)

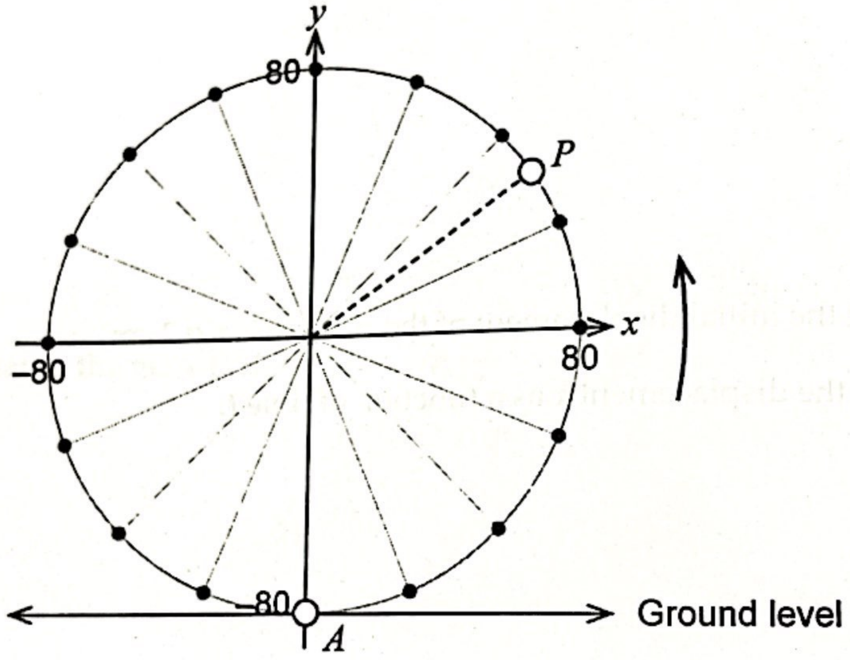
4. **[11 marks]**

A ferris wheel has a radius of 80 metres and rotates in an anticlockwise direction at a rate of one revolution every 72 seconds. The ferris wheel has 16 cars that are equally spaced around the wheel as shown in the diagram.

A coordinate system is set up so that the centre of the ferris wheel is at the origin and the ground level has equation $y = -80$. Passengers begin their ride when a car is at position A $(0,-80)$.

Consider a passenger in a car at position P.

Let t = the number of seconds the ride has been in progress from position A.
θ = the angle in radians that the car has rotated from position A.
y = the height of a car above the centre of the ferris wheel (metres).



(a) Show that $\frac{d\theta}{dt} = \frac{\pi}{36}$ radians per second. [1]

(b) Given that $y(\theta) = 80\sin(\theta + a)$, explain why $a = -\frac{\pi}{2}$ [1]

4. (cont)

(c) Determine how quickly a passenger is moving upward when they are 100 metres above the ground, correct to the nearest 0.01 metres per second. [4]

(d) Show that function $y(t)$ satisfies the condition for simple harmonic motion. [2]

A different passenger happens to be in a car that is two cars ahead of a particular car on the ferris wheel.

(e) At what speed, correct to the nearest 0.01 metres per second, is the trailing passenger moving upward when the other passenger is moving downward at exactly the same speed? [3]

(MSPEC 2020:CA14)

5. **[5 marks]**

A particle travels in a straight line so that its velocity v cm per second and displacement x cm are related by the equation:

$$v = -0.2x$$

(a) Determine the acceleration a in terms of its displacement x. [2]

(b) Does the particle's motion constitute simple harmonic motion? Justify your answer. [1]

It is known that the initial displacement of the particle is $x = 4$ cm.

(c) Determine, correct to the nearest 0.01 second, when the particle has a displacement of 2 cm. [2]

6. **[6 marks]**

(MSPEC 2021:CA12)

The horizontal displacement of a Ferris wheel cabin exhibits simple harmonic motion. The maximum horizontal speed is $\frac{\pi}{2}$ metres per second and its period of motion is exactly 60 seconds.

Let $x(t) = A\cos(nt)$ be the horizontal displacement after t seconds.

(a) Determine the values of A and n. [3]

(b) Determine the horizontal acceleration, correct to the nearest 0.001 m/s^2, when the horizontal displacement is 10 metres. [3]

Chapter

18

Sample Means and Probability Distributions

(3CDMAT 2010:CA18a,d)

[3 marks]

he burn time, T seconds, of a randomly-chosen match produced by the Ever-Flame company normally distributed, with a mean of 12.2 seconds and a standard deviation of 2.5 seconds.

a) Calculate $P(T > 16)$ [1]

b) Every week the company tests its matches by measuring the burn times of 1000 randomly-chosen matches.

What is the probability that the average burn time of the matches in such a sample will be between 12.15 and 12.25 seconds? [2]

2. **[12 marks]** *(MSPEC 2016:CA19a,b,c,d)*

The volume of water used by the SavaDaWater company to top up an ornamental pool has been observed to be normally distributed with mean $\mu = 175$ litres and standard deviation $\sigma = 15$ litres.

The ornamental pool is topped up 50 times. Determine the probability that the:

(a) sample mean volume will be between 173 and 177 litres. [3]

(b) total volume of water used is less than 8.96 kilolitres. [3]

Water is a scarce commodity and accuracy is required. The pool is topped up 50 times and the sample mean obtained is denoted by $\overline{W}$.

(c) If it is required that $P(a \leq \overline{W} \leq b) = 0.99$, then determine the values of a and b, each correct to 0.1 litres. [3]

2. (cont)

(d) If the probability for the mean amount of water used differs from μ by less than five litres is 96%, find n, the number of waterings that need to be measured. [3]

3. **[6 marks]** *(MSPEC 2017:CA9)*

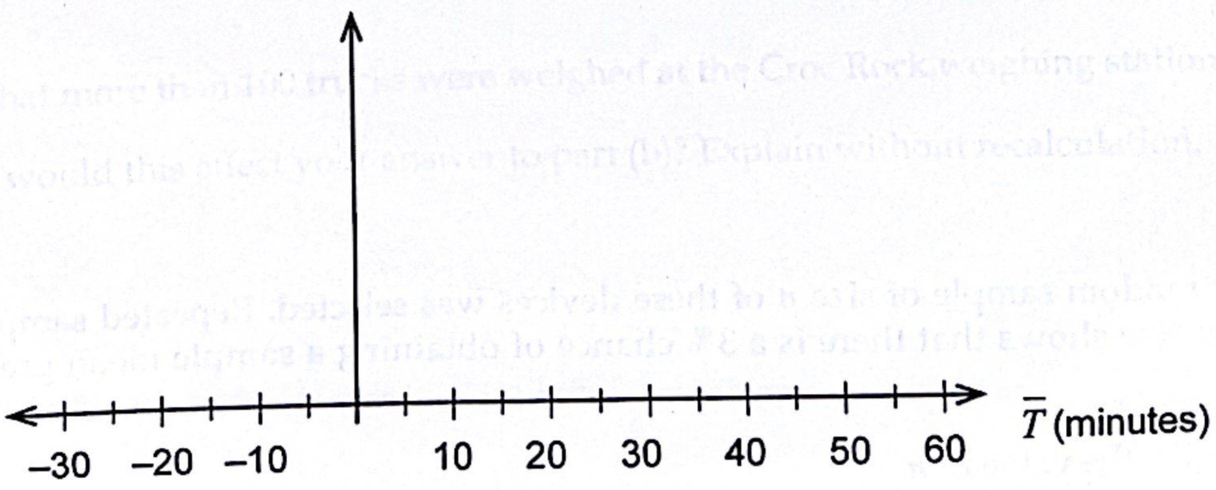
The time T in minutes that a particular flight arrives later than its scheduled time is uniformly distributed with $-30 \leq T \leq 60$. The population mean is $\mu\,(T) = 15$ and the population variance is $\sigma^2\,(T) = 675$.

A sample of 30 arrival times is taken and the sample mean $\overline{T}$ is calculated.

(a) Determine $P\,(10 \leq \overline{T} \leq 20)$ correct to 2 decimal places. [3]

(b) If a large number of samples, each with 30 arrival times, is taken, sketch the likely distribution of the sample mean $\overline{T}$ below.

In the diagram indicate or refer to the calculation from part (a). [3]



(MSPEC 2018:CA12)

4. **[11 marks]**

The lifetime of an electronic device is distributed as an exponential random variable with mean $\mu = 20$ years and standard deviation $\sigma = 20$ years. A random sample of 50 of these devices is selected. Tam, a graduate electronics engineer, is interested in the sample mean lifetime $\overline{X}$ of these 50 devices.

(a) State the distribution of the sample mean lifetime $\overline{X}$. Justify your answer. [3]

(b) Determine the probability that the sample mean lifetime is between 15 and 25 years. [2]

Jai, the chief engineer, informs Tam that the lifetimes may not be exponentially distributed but could be a more complicated distribution, yet still having mean $\mu = 20$ years and standard deviation $\sigma = 20$ years.

(c) If Jai is correct, will your answer to part (b) change? Explain. [2]

A different random sample of size n of these devices was selected. Repeated sampling with this sample size shows that there is a 3% chance of obtaining a sample mean greater than 25 years.

(d) Determine the value of n. [4]

5. **[10 marks]**

(MSPEC 2019:CA14)

Trucks carrying iron ore for the Croc Rock mining company arrive at a weighing station. The service time T per truck is defined to be the time elapsed from the moment a truck enters the station zone, including the time to be positioned and then weighed, up to the time it leaves the zone.

It is known that the population mean $\mu(T) = 80$ seconds and the population standard deviation $\sigma(T) = 20$ seconds.

At the Croc Rock weighing station, 100 trucks are weighed.

(a) State the (approximate) distribution of the sample mean service time per truck for the 100 trucks. [3]

(b) What is the probability that the sample mean service time will be more than 83 seconds? [2]

Suppose that more than 100 trucks were weighed at the Croc Rock weighing station.

(c) How would this affect your answer to part (b)? Explain without recalculation. [2]

It is desired that the probability that the sample mean service time will be between 80 seconds and 82 seconds is greater than 40%.

(d) Determine the minimum number of trucks that will need to be weighed. [3]

(MSPEC 2020:CA18a,b)

6. **[7 marks]**

The mass of chocolate that is placed into each biscuit produced by the BikkiesAreUs company has been observed to be normally distributed with mean μ = 7.5 grams and standard deviation $\sigma = 1.5$ grams.

(a) Determine the probability, correct to 0.01, that the total amount of chocolate used for 50 biscuits is less than 365 grams. [4]

(b) If the probability that the mean amount of chocolate used per biscuit differs from μ by less than 0.2 grams is 98%, determine n, the number of biscuits that need to be sampled. [3]

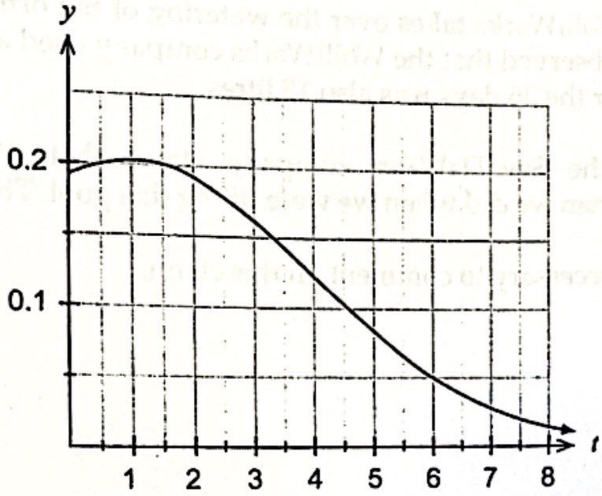
7. **[5 marks]**

(MSPEC 2021:CA15a,b)

An experiment was conducted to measure how quickly adults respond to the request: 'send me a text message'.

Let T = the number of hours taken for an adult to respond and send a text message.

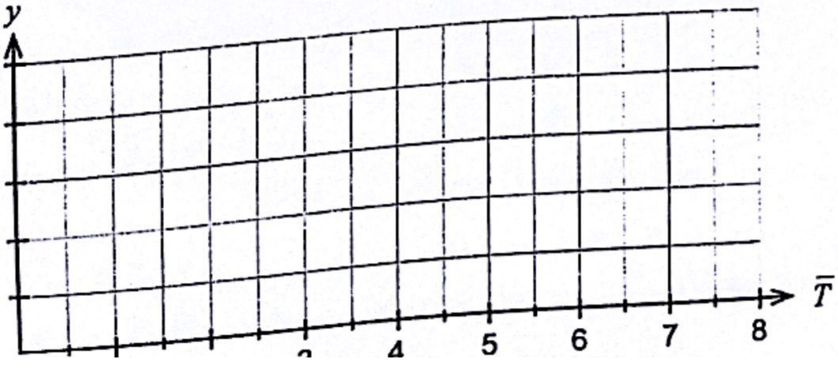
It was found that the distribution of the population of response times for adults was given by the probability density function shown below, with mean $\mu = 3$ hours and standard deviation $\sigma = 2.4$ hours.



Random samples of size 64 are drawn repeatedly from the population of response times and the sample mean response time $\bar{T}$ is determined for each sample.

(a) Calculate, correct to 0.001, the probability that a sample mean response time will be between 150 minutes and 210 minutes. [3]

(b) Sketch the likely distribution of the sample mean $\bar{T}$ (for samples of size 64) on the axes below. [2]



Chapter 19

Sample Means and Confidence Intervals

(MSPEC 2016:CA19e)

1. **[4 marks]**

The volume of water used by the SavaDaWater company to top up an ornamental pool has been observed to be normally distributed with mean $\mu = 175$ litres and standard deviation $\sigma = 15$ litres.

A rival company called WolliWorks takes over the watering of the ornamental pool. Over 36 consecutive days, it was observed that the WolliWorks company used a total of 6.57 kilolitres. The standard deviation for the 36 days was also 15 litres.

A representative from the SavaDaWater company states that 'WolliWorks are using significantly more water than we did when we were filling this pool. They are wasting water'.

Perform the calculations necessary to comment on this claim.

2. **[13 marks]** *(MSPEC 2017:CA13)*

A cable in a bridge is required to support a weight of 10,000 Newtons. Tina tests a random sample of 100 cables from a supplier. The sample mean is found to be 10,300 Newtons and the sample standard deviation 400 Newtons.

(a) Based on Tina's sample, obtain a 95% confidence interval for μ, the population mean cable strength. **[4]**

(b) State whether each of the following statements is true or false. Provide reasons for your answer and state any assumptions.

(i) If another sample of 100 cables is taken, then the sample mean will fall within the confidence interval found at part (a). **[2]**

(ii) If a single cable is selected at random, then the strength of the cable will fall within the confidence interval found at part (a). **[2]**

2. (cont)

Jon, a colleague of Tina, said, 'The cable strengths are not normally distributed, so the calculation for the confidence interval is incorrect'.

(c) How should Tina respond to Jon's comment? [2]

A different sample of 36 cables is taken and it is found that the standard deviation is 500 Newtons. A confidence interval for the population mean cable strength is determined to be $9900 \le \mu \le 10{,}200$.

(d) Determine the confidence level, to the nearest 0.1%, used to calculate this interval. [3]

3. **[9 marks]**

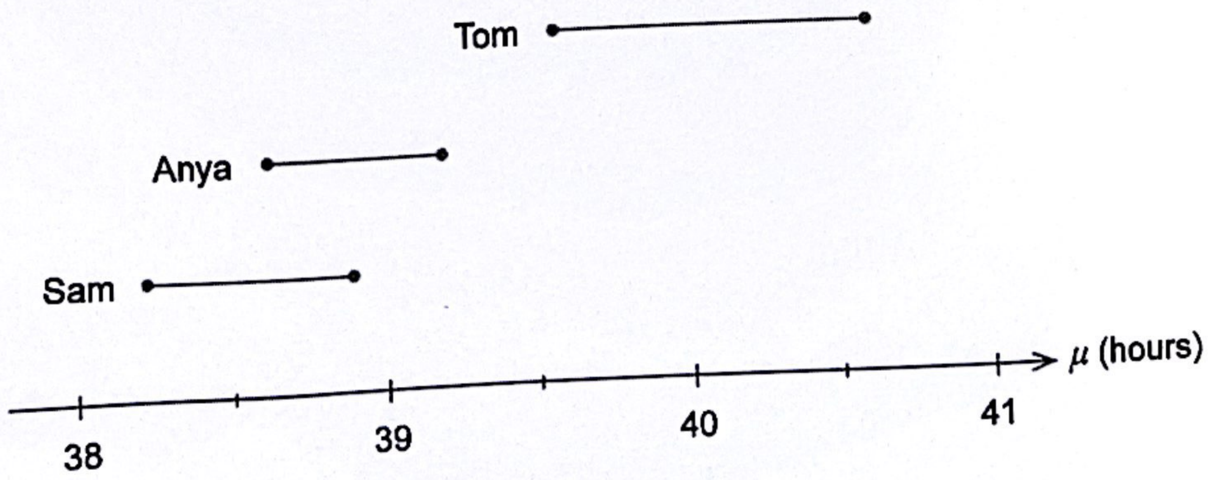
(MSPEC 2018:CA16)

Tom wants to estimate the population mean number of hours, μ, worked by Australians per week. He takes a random sample of 400 workers and determines a 99% confidence interval for μ. The upper limit of this interval is 40.62 hours and the width of this interval is 1.08 hours.

(a) Determine the sample mean for this sample of 400 workers. [2]

(b) Calculate, correct to 0.01 hours, the sample standard deviation for the sample of 400 workers. [3]

Two of Tom's colleagues, Anya and Sam, each take different random samples of size 400 and similarly determine 99% confidence intervals for the population mean μ. These confidence intervals, together with Tom's, are shown below.



3. (cont)

(c) Anya makes the following statements based on these confidence intervals. Indicate why each of her statements is true or false.

(i) 'Tom's sample has a larger standard deviation compared with that of Sam's and mine.' [1]

(ii) 'Tom's method of sampling must be biased since his confidence interval does not overlap with mine or Sam's.' [1]

(d) Which of these three confidence intervals contains the value for μ? Justify your answer. [2]

4. **[9 marks]** *(MSPEC 2019:CA15)*

A random sample of n commuters in Melbourne in August 2018 found that the average time to commute to work was 40 minutes. Repeated sampling of the mean indicated that the standard deviation of the sample mean was 3 minutes.

(a) Determine a 90% confidence interval for the population mean commuting time μ to work, correct to 0.01 minutes. [3]

Another random sample of $2n$ commuters in November 2018 found that the average time to commute to work was 45 minutes. Assume that both the August and November samples were drawn from the same population.

(b) What is the standard deviation of the sample mean for the November sample, correct to 0.01 minutes? [2]

4. (cont)

Suppose that the August and November samples are combined to form a sample with $3n$ commuters. Consider 90% confidence intervals for the following samples for the purpose of determining the population mean commuting time μ.

90% confidence interval	Sample	Size
A	August	n
N	November	$2n$
C	Combined	$3n$

(c) Which of the three confidence intervals, A, N or C, will provide the greatest precision in determining the population mean μ? Justify your answer. [2]

(d) Which of the three confidence intervals, A, N or C, contains the true value of the population mean μ? Justify your answer. [2]

5. **[8 marks]** *(MSPEC 2020:CA17)*-

Members of a random sample of n shoppers at the El Cheepo shopping centre were asked by a consumer researcher how much they had spent in the shopping centre that day. Let μ denote the mean and σ the standard deviation of the amount spent. The standard deviation σ is known from previous research.

A 95% confidence interval for μ based on the sample is $150 \leq \mu \leq 200$ dollars.

(a) Determine the sample mean for this sample. [1]

(b) Based on this confidence interval, calculate the standard deviation of the sample mean, correct to 0.01. [3]

The following week, the researcher again took a random sample of shoppers from the El Cheepo shopping centre, but this time the sample size was doubled.

(c) What is the probability that the difference between μ and the sample mean from this sample will be less than $10? [4]

6. **[4 marks]** *(MSPEC 2020:CA18c)*

The mass of chocolate that is placed into each biscuit produced by the BikkiesAreUs company has been observed to be normally distributed with mean μ = 7.5 grams and standard deviation $\sigma = 1.5$ grams.

A competitor company called YouBeautChokkies produces similar biscuits. A sample of 144 biscuits was taken and it was found that the standard deviation of the mass of chocolate used in each biscuit was 1.8 grams and the total amount of chocolate used in the sample of 144 biscuits was 1.09 kg.

Charlie Chokka, a representative from the YouBeautChokkies company, stated that 'we are using significantly more chocolate for each biscuit than BikkiesAreUs. If you want that real chocolate taste, then buy from us!'

Perform the necessary calculations to comment on Charlie's claim.

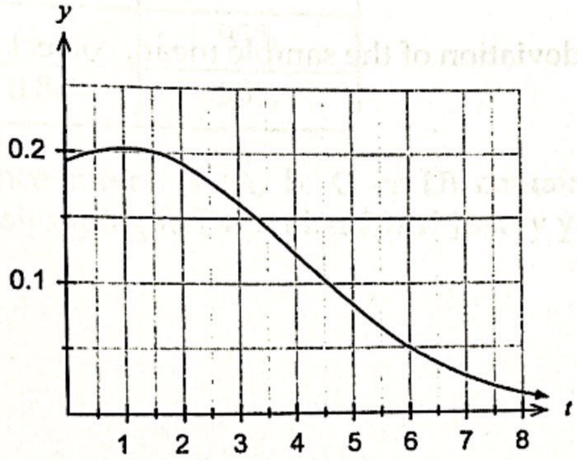
7. **[4 marks]**

(MSPEC 2021:CA15c)

An experiment was conducted to measure how quickly adults respond to the request: 'send me a text message'.

Let T = the number of hours taken for an adult to respond and send a text message.

It was found that the distribution of the population of response times for adults was given by the probability density function shown below, with mean $\mu = 3$ hours and standard deviation $\sigma = 2.4$ hours.



Anika, a teacher at the TekNoCrat School, theorises that as teenagers tend to check their text messages more frequently than adults, then the population mean response time for teenagers will be much lower than the population mean adult response time $\mu = 3$.

Anika is then presented with the sample mean response time for a sample gathered from an unknown source.

Sample size	Sample mean (hours)	Sample standard deviation (hours)
100	2.1	2.7

Calculations are performed and Anika concludes by stating: 'this sample was clearly not taken from the population of adult response times. It is highly likely that this sample was taken from a sample of 100 teenagers'.

Perform the necessary calculations and comment on Anika's claim.

8. **[12 marks]** *(MSPEC 2021:CA17)*

A researcher is interested in estimating the population mean μ (dollars) that Perth residents had spent via online shopping in December 2020. A random sample of size n gave a sample mean of \$400, a sample standard deviation s and a 95% confidence interval of width \$200.

(a) State the 95% confidence interval obtained. [1]

(b) Calculate the standard deviation of the sample mean, correct to \$0.01. [2]

(c) In terms of n, what sample size would yield a 95% confidence interval of width \$50? Show your reasoning. [2]

(d) What is the probability that another sample of size $2n$ would produce a sample mean that differs from μ by more than \$50? [3]

8. (cont)

Four different confidence intervals (A, B, C and D) are obtained for the mean amount spent via online shopping by Perth residents in December 2020.

Confidence interval	Sample size	Sample standard deviation	Confidence level
A	n	s	95%
B	n	s	99%
C	$2n$	s	95%
D	n	$0.8s$	95%

(e) Which of the confidence intervals (A, B, C or D) contains μ, the population mean expenditure for online shopping in December 2020? Justify your answer. [2]

(f) For each of the following, state the confidence interval that has the smaller width. Justify your answers.

(i) A and B. [1]

(ii) C and D. [1]

Chapter 1: Cartesian and Polar Forms

1. (MSPEC 2016:CF02)

(a) $\frac{\overline{2+i}}{(1-i)^2}$

$= \frac{2-i}{-2i} \circ \frac{i}{i}$

$= \frac{2_i+1}{2}$

$= \frac{1}{2} + i$

(b) $(\sqrt{3}-\mathrm{i})^5$

$= \left(2\mathrm{cis}\left(-\frac{\pi}{6}\right)\right)^5$

$= 32\ \mathrm{cis}\left(-\frac{5\pi}{6}\right)$

$= 32\left(\frac{-\sqrt{3}}{2} - \frac{i}{2}\right)$

$= -16\sqrt{3} - 16i$

2. (MSPEC 2017:CF1)

$w = a - bi + i\,(a + bi)$
$= a - bi + ai - b$
$= a - b + (a - b)\,i$

$\arg(w) = \tan^{-1}\left(\frac{a-b}{a-b}\right)$

$= \tan^{-1}(1)$

$= \frac{\pi}{4}, \frac{-3\pi}{4}$

Note for $a = b$ $w = 0 + 0i$ and $\arg(w)$ is undefined
Alternative solution is:

Consider graphically by drawing the complex vectors z (4th quadrant), $i\bar{z}$ (2nd quadrant) and the resultant $z + i\bar{z}$ which will be in the first quadrant if $a > b$ or in the third quadrant if $a < b$. Since the sides of the vector triangle given by z and $i\bar{z}$ are congruent, we have an isosceles triangle. Using this and alternate angles it can be shown that the two possible answers for $\arg(w)$ are 45° and 135°

3. (MSPEC 2017:CA10)

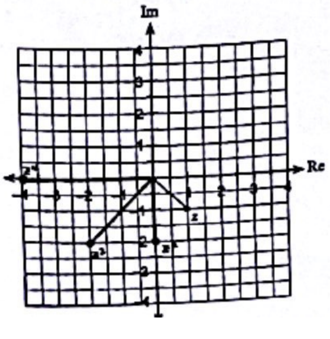
(a) $\sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$

(b) $z^2 = \left(\sqrt{2}\mathrm{cis}\left(-\frac{\pi}{4}\right)\right)^2 = 2\mathrm{cis}\left(-\frac{\pi}{2}\right)$

$z^3 = 2\sqrt{2}\mathrm{cis}\left(\frac{-3\pi}{4}\right)$

$z^4 = 4\ \mathrm{cis}(-\pi)$

(c)



(d) Expansion by a factor of $\sqrt{2}$ and rotation of $\frac{\pi}{4}$ clockwise.

4. (MSPEC 2018:CF3)

(a) $\mathrm{Re}\left[\frac{a+(b-1)i}{a+bi} \cdot \frac{a-bi}{a-bi}\right] = 0$

$$\frac{a^2 + b(b-1)}{a^2+b^2} = 0$$

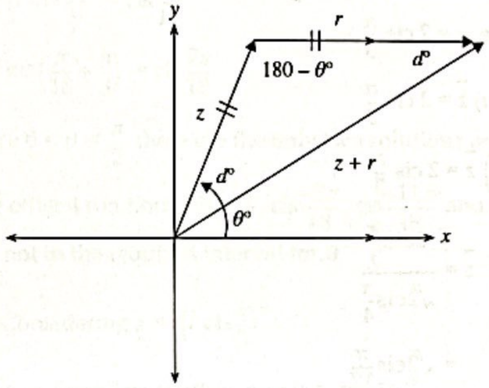
No need to consider Imaginary part
$a^2 + b^2 - b = 0$
$a^2 = b - b^2$

(b) (i) $\frac{2\mathrm{cis}\frac{\pi}{2}}{r\mathrm{cis}(-\theta)}$

$\frac{2}{r}\mathrm{cis}\left(\frac{\pi}{2} - -\theta\right)$

$\frac{2}{r}\mathrm{cis}\left(\theta + \frac{\pi}{2}\right)$

(ii)



Consider $z + r$ as the resultant of the two complex vectors. Since the triangle is isosceles with equal sides of 'r' it follows that $d = \frac{\theta}{2}$ and so $\arg(z + r) = \frac{\theta}{2}$

5. (MSPEC 2019:CA12)

(a) $r = \sqrt{\left(\frac{1}{2\sqrt{2}}\right)^2 + \left(\frac{-1}{2\sqrt{2}}\right)^2} = \frac{1}{2}$

$\theta = \tan^{-1}\left(\frac{\frac{-1}{2\sqrt{2}}}{\frac{1}{2\sqrt{2}}}\right) = \frac{-\pi}{4}$

$\therefore w = \frac{1}{2}\ \mathrm{cis}\left(\frac{-\pi}{4}\right)$

(b) $z = 4\ \mathrm{cis}\ \frac{8\pi}{12} = 4\ \mathrm{cis}\ \frac{2\pi}{3}$

$= -2 + 2\sqrt{3}i$

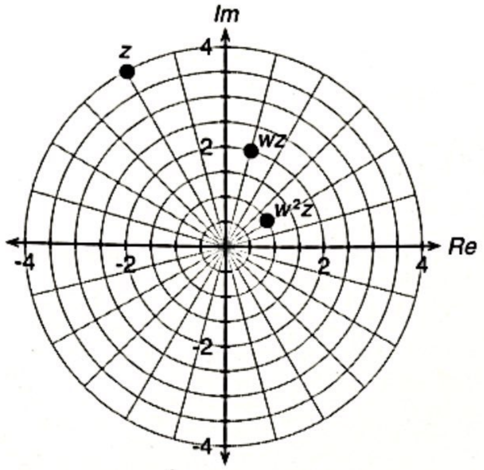
(c) $wz = \frac{1}{2} \cdot 4\ \mathrm{cis}\left(\frac{-\pi}{4} + \frac{2\pi}{3}\right) = 2\ \mathrm{cis}\ \frac{5\pi}{12}$

$w^2z = \left(\frac{1}{2}\mathrm{cis}\left(\frac{-\pi}{4}\right)\right)^2 \cdot 4\ \mathrm{cis}\ \frac{2\pi}{3}$

$= \frac{1}{4} \cdot 4\ \mathrm{cis}\left(\frac{-2\pi}{4} + \frac{2\pi}{3}\right)$

$= \mathrm{cis}\ \frac{\pi}{6}$

(d)



(e) Dilation factor $\frac{1}{2}$ (contraction) and rotation clockwise of $\frac{\pi}{4}$

6. (MSPEC 2020:CA11)

(a) Since $w = 2 \text{ cis } \frac{\pi}{3}$

and $(1 + i)\,\bar{z} = 2 \text{ cis } \frac{\pi}{3}$

$\left(\sqrt{2}\text{cis}\frac{\pi}{4}\right)\bar{z} = 2 \text{ cis } \frac{\pi}{3}$

$$\bar{z} = \frac{2\text{cis}\frac{\pi}{3}}{\sqrt{2}\text{cis}\frac{\pi}{4}}$$

$$= \sqrt{2}\text{cis}\frac{\pi}{12}$$

$$\therefore z = \sqrt{2}\text{cis}\left(\frac{-\pi}{12}\right)$$

$$u = \frac{\sqrt{2}\text{cis}\left(\frac{-\pi}{12}\right)}{2-2i} = \frac{\sqrt{2}\text{cis}\left(\frac{-\pi}{12}\right)}{2(1-i)} = \frac{\sqrt{2}\text{cis}\left(\frac{-\pi}{12}\right)}{2\left(\sqrt{2}\text{cis}\left(\frac{-\pi}{4}\right)\right)}$$

$$= \frac{1}{2}\text{cis }\left(\frac{-\pi}{12} - \frac{-\pi}{4}\right)$$

$$= \frac{1}{2}\text{cis }\left(\frac{\pi}{6}\right)$$

$$\therefore \text{Arg }(u) = \left(\frac{\pi}{6}\right)$$

(b) from above $|\,u\,| = \frac{1}{2}$

7. (MSPEC 2020:CA15)

(a) $= \dfrac{r\,cis(-\theta)}{-\sqrt{2}(\sqrt{2}\,cis\frac{\pi}{4})}$

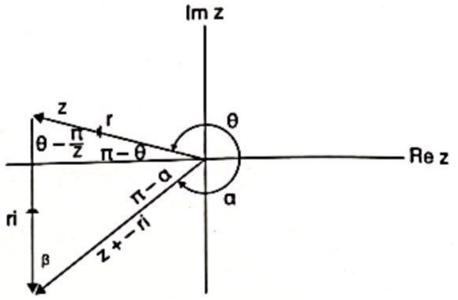
$= -\frac{r}{2}cis(-\theta - \frac{\pi}{4})$

$= cis\,\pi \cdot \frac{r}{2}\,cis\,(-\theta - \frac{\pi}{4})$

$= \frac{r}{2}\,cis\,(-\theta - \frac{\pi}{4} + \pi)$

$= \frac{r}{2}\,cis\,(-\theta + \frac{3\pi}{4})$

(b)



$$\text{Angle } \beta = \frac{\pi - \left(\theta - \frac{\pi}{2}\right)}{2} \text{ from diagram above}$$

$$= \frac{3\pi}{4} - \frac{\theta}{2}$$

Since triangle above is isosceles

$\pi - \alpha + \pi - \theta = \frac{3\pi}{4} - \frac{\theta}{2}$ (Considering positive angles only with no direction)

$2\pi - \frac{3\pi}{4} - \theta + \frac{\theta}{2} = \alpha$

$$\alpha = \frac{5\pi}{4} - \frac{\theta}{2}$$

Considering clockwise rotation for α

$\alpha = -\left(\frac{5\pi}{4} - \frac{\theta}{2}\right)$

$\alpha = \frac{\theta}{2} - \frac{5\pi}{4}$

$\alpha = \frac{\theta}{2} + \frac{3\pi}{4}$

8. (MSPEC 2021:CF1)

(a) $-\frac{2\pi}{3}$

(b) $\left|\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} + 2\right|$

$\left|2.5 + \frac{\sqrt{3}}{2}i\right|$

$\sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$\sqrt{7}$

9. (MSPEC 2021:CA11a,b,c)

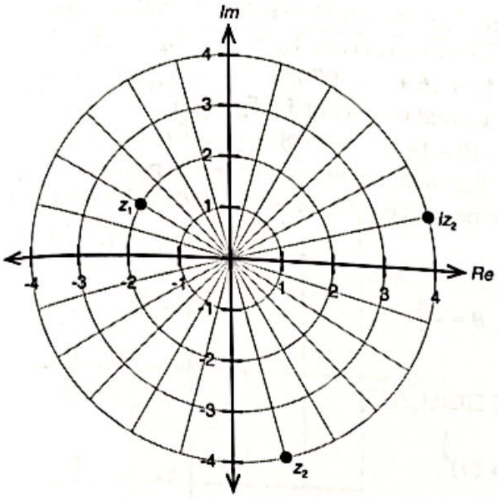
(a) $2\text{ cis}\frac{5\pi}{6}$

(b) $z_1 = 2\cos\frac{5\pi}{6} + 2i\sin\frac{5\pi}{6}$

$= 2\left(-\frac{\sqrt{3}}{2}\right) + 2i\left(\frac{1}{2}\right)$

$= -\sqrt{3} + i$

(c)



Chapter 2: Powers and Roots of Complex Numbers

1. (3CDMAS 2015:CF8)

(a) $k = (2\text{ cis}(-36°))^5$
$= 32\text{ cis }(-180°)$
$= -32$

(b) Roots are 72° apart
$2e^{ix} = 2\text{ cis }108°$

$108° = \frac{3\pi}{5}$

$x = \frac{3\pi}{5}$

(c) $z_2 = 2\text{ cis }108°$
$\bar{z}_2 = 2\text{ cis }(-108°)$
$z_4 = 2\text{ cis }(-108°)$
$z_4 = \bar{z}_2$

(d) Given

$$2\text{cis}\frac{\pi}{5} + 2\text{cis}\frac{3\pi}{5} + 2\text{cis}\frac{5\pi}{5} + 2\text{cis}\frac{7\pi}{5} + 2\text{cis}\frac{9\pi}{5} = 0$$

$$2\text{cis}\frac{\pi}{5} + 2\text{cis}\frac{3\pi}{5} + 2\text{cis}\frac{7\pi}{5} + 2\text{cis}\frac{9\pi}{5} = 2$$

$$\text{cis}\frac{\pi}{5} + \text{cis}\frac{3\pi}{5} + \text{cis}\frac{7\pi}{5} + \text{cis}\frac{9\pi}{5} = 1$$

$$\text{cis}\frac{\pi}{5} + \left(\text{cis}\frac{\pi}{5}\right)^3 + \left(\text{cis}\frac{\pi}{5}\right)^7 + \left(\text{cis}\frac{\pi}{5}\right)^9 = 1$$

$$w + w^3 + w^7 + w^9 = 1$$

2. (MSPEC 2016:CA14)

(a) $z^4 = 16\text{ cis }\left(\frac{-\pi}{2}\right)$

$$z = \left(16\text{cis}\left(\frac{-\pi}{2}\right)\right)^{\frac{1}{4}}$$

$$z_1 = 2\text{ cis }\left(\frac{-\pi}{8}\right)$$

$$z_2 = 2\text{ cis }\left(\frac{3\pi}{8}\right)$$

$$z_3 = 2\text{ cis }\left(\frac{-5\pi}{8}\right)$$

$$z_4 = 2\text{ cis }\left(\frac{7\pi}{8}\right)$$

(b) $\arg\left(2\text{cis}\left(\frac{3\pi}{8}\right) + 2\right)$

from Classpad $\arg(w + 2) = 0.5890$ or $\frac{3\pi}{16}$

3. (MSPEC 2017:CA19)

(a) $z^6 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

$$\tan\theta = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

$$\theta = \frac{\pi}{3}$$

All roots are $\frac{2\pi}{6\text{ roots}} = \frac{\pi}{3}$ apart

$$z_1 = \left(1\text{ cis}\frac{\pi}{3}\right)^{\frac{1}{6}} = \text{cis}\frac{\pi}{18}$$

$$z_2 = \text{cis}\left(\frac{\pi}{18} + \frac{\pi}{3}\right) = \text{cis}\frac{7\pi}{18}$$

Since $0 < \theta < \frac{\pi}{2}$ these are the only two solutions required.

The other 4 solutions, $\text{cis}\frac{13\pi}{18}$, $\text{cis}\frac{-5\pi}{18}$, $\text{cis}\frac{-11\pi}{18}$ and $\text{cis}\frac{-17\pi}{18}$ are not in the required interval for θ.

(b) Considering $z = \left(1\text{ cis}\frac{\pi}{3}\right)^{\frac{1}{n}}$

First and smallest positive root is $z_1 = \text{cis}\frac{\pi}{3n}$

Since exactly 3 roots in the first quadrant then the third root will be $z_3 = \text{cis }\left(\frac{\pi}{3n} + 2 \cdot \frac{2\pi}{n}\right)$

$$= \text{cis }\left(\frac{\pi}{3n} + \frac{4\pi}{n}\right)$$

The fourth root (in order) should be outside the first quadrant but will be given by $z_4 = \text{cis }\left(\frac{\pi}{3n} + 3 \cdot \frac{2\pi}{n}\right)$

$$= \text{cis }\left(\frac{\pi}{3n} + \frac{6\pi}{n}\right)$$

Since the first quadrant has $0 < \theta < \frac{\pi}{2}$ then we require

$$\frac{\pi}{3n} + \frac{4\pi}{n} < \frac{\pi}{2}$$

$$\frac{13\pi}{3n} < \frac{\pi}{2}$$

$$n > \frac{26}{3} \quad \text{i.e. } n \geq 9$$

and also

$$\frac{\pi}{3n} + \frac{6\pi}{n} > \frac{\pi}{2}$$

$$\frac{19\pi}{3n} > \frac{\pi}{2}$$

$$n < \frac{38}{3} \quad \text{i.e. } n \leq 12$$

So $9 \leq n \leq 12$, n an integer

4. (MSPEC 2018:CA10)

$$\text{LHS} = \left(2\text{cis}\frac{\pi}{6}\right)^n - \left(2\text{cis}\left(\frac{-\pi}{6}\right)\right)^n$$

$$= 2^n\,\text{cis}\left(\frac{n\pi}{6}\right) - 2^n\,\text{cis}\left(-\frac{n\pi}{6}\right)$$

$$= 2^n\left[\cos\left(\frac{n\pi}{6}\right) + i\sin\left(\frac{n\pi}{6}\right) - \cos\left(-\frac{n\pi}{6}\right) - i\sin\left(-\frac{n\pi}{6}\right)\right]$$

$$= 2^n\left[\cos\left(\frac{n\pi}{6}\right) + i\sin\left(\frac{n\pi}{6}\right) - \cos\left(\frac{n\pi}{6}\right) + i\sin\left(\frac{n\pi}{6}\right)\right]$$

$$= 2^n\left[2i\sin\left(\frac{n\pi}{6}\right)\right]$$

$$= 2^{n+1}\sin\left(\frac{n\pi}{6}\right)i$$

$$= \text{RHS}$$

5. (MSPEC 2019:CF9)
Investigate with $n = 3, 4, 5, 6$ for pattern

$n = 3 \quad p = (\text{cis}\,0)\left(\text{cis}\,\frac{2\pi}{3}\right)\left(\text{cis}\left(\frac{-2\pi}{3}\right)\right) = 1$

$n = 4 \quad p = (\text{cis}\,0)\left(\text{cis}\,\frac{\pi}{2}\right)\left(\text{cis}\left(\frac{-\pi}{2}\right)\right)(\text{cis}\,\pi) = -1$

$n = 5 \quad p = (\text{cis}\,0)\left(\text{cis}\,\frac{2\pi}{5}\right)\left(\text{cis}\left(\frac{-2\pi}{5}\right)\right)\left(\text{cis}\frac{4\pi}{5}\right)\left(\text{cis}\left(\frac{-4\pi}{5}\right)\right) = 1$

$n = 6 \quad p = (\text{cis}\,0)\left(\text{cis}\,\frac{2\pi}{6}\right)\left(\text{cis}\left(\frac{-2\pi}{6}\right)\right)\left(\text{cis}\frac{4\pi}{6}\right)\left(\text{cis}\left(\frac{-4\pi}{6}\right)\right)$

$\left(\text{cis}\,\frac{6\pi}{6}\right) = -1$

$\therefore p = 1$ if n is odd
$p = -1$ if n is even

<u>Alternative solution</u>

n even $\Rightarrow$ $p = \text{cis}\,0 \,.\, \text{cis}\,\frac{2\pi}{n} \,.\, \text{cis}\left(\frac{-2\pi}{n}\right)\text{cis}\left(\frac{4\pi}{n}\right)\text{cis}\left(\frac{-4\pi}{n}\right)\ldots$

$\text{cis}\left(\frac{n\pi}{n}\right) = \text{cis}\,0 \,.\, \text{cis}\,\pi = -1$

n odd $\Rightarrow$ $p = \text{cis}\,0 \,.\, \text{cis}\,\frac{2\pi}{n} \,.\, \text{cis}\left(\frac{-2\pi}{n}\right)\ldots\text{cis}\left(\frac{(n-1)\pi}{n}\right)$

$\text{cis}\left(\frac{-(n-1)\pi}{n}\right) = \text{cis}\,0 = 1$

6. (MSPEC 2020:CF8)
Since $i = 1$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$
$T_1, T_5, T_9, T_{13}, \ldots$ will be $+i$ terms (i.e. $i^5 = i^9 = i$)
So $i + 5i + 9i + 13i + \ldots 2017i$
$i\,(1 + 5 + 9 + 13 + \ldots 2017)$

$T_3, T_7, T_{11}, T_{15}, T_{19} \Rightarrow$ sum of these is
$-3i - 7i - 11i - 15i - \ldots -2019i$
$-i\,(3 + 7 + 11 + 15 + \ldots 2019)$
The negative i terms outweigh the positive i terms by $-2i$ on each term.

There are $\frac{2020}{4} = 505$ terms for both.

So the sum of all the 'i' terms is $505 \times (-2i) = -1010i$.
The positive real terms are $T_4, T_8, T_{12}, T_{16}, \ldots T_{2020}$
$4 + 8 + 12 + 16 + \ldots + 2020.$
The negative real terms are $T_2, T_6, T_{10}, T_{14}, \ldots T_{2018}$
$-2 - 6 - 10 - 14 \ldots - 2018$
Summing the real terms where each positive term outweighs the negative by 2. So sum is $505 \times 2 = 1010$ $\therefore$ Total sum $= 1010 - 1010i$ therefore $r\,\text{cis}\theta$ has $r = \sqrt{1010^2 + 1010^2}$ $= 1010\sqrt{2}$, $\theta = -\frac{\pi}{4}$.

7. (MSPEC 2020:CA13)

$z = (8\sqrt{3} + 8i)^{\frac{1}{4}}$

$z_1 = \left(16\,\text{cis}\frac{\pi}{6}\right)^{\frac{1}{4}} \qquad r = \sqrt{(8\sqrt{3})^2 + 8^2}$

$= 2\,\text{cis}\,\frac{\pi}{24} \qquad\qquad = 16$

$= Arg\,z = \tan^{-1}\left(\frac{8}{8\sqrt{3}}\right) = \frac{\pi}{6}$

$z_2 = 2\,\text{cis}\left(\frac{\pi}{24} + \frac{\pi}{2}\right) = 2\,\text{cis}\left(\frac{13\pi}{24}\right)$

$z_3 = 2\,\text{cis}\left(\frac{\pi}{24} - \frac{\pi}{2}\right) = 2\,\text{cis}\left(\frac{-11\pi}{24}\right)$

$z_4 = 2\,\text{cis}\left(\frac{\pi}{24} + \pi\right) = 2\,\text{cis}\,\frac{25\pi}{24} = 2\,\text{cis}\left(\frac{-23\pi}{24}\right)$

8. (MSPEC 2021:CF7)

(a) $z = (1)^{\frac{1}{43}}$

$z = (\text{cis}(0 + 2k\pi))^{\frac{1}{43}}$, $k = 0, 1, 2, \ldots 42$

$z = \text{cis}\frac{2k\pi}{43}$

$z = \text{cis}0, \text{cis}\frac{2\pi}{43}, \text{cis}\frac{4\pi}{43}, \ldots \text{cis}\frac{84\pi}{43}$

(b) Solutions the same when

$\text{cis}\left(\frac{2k\pi}{43}\right) = \text{cis}\left(\frac{2k\pi}{47}\right)$

$\frac{2k\pi}{43} = \frac{2k\pi}{47}$

$\frac{k}{43} - \frac{k}{47} = 0$

$\frac{47k - 43k}{(43)(47)} = 0$

$4k = 0$

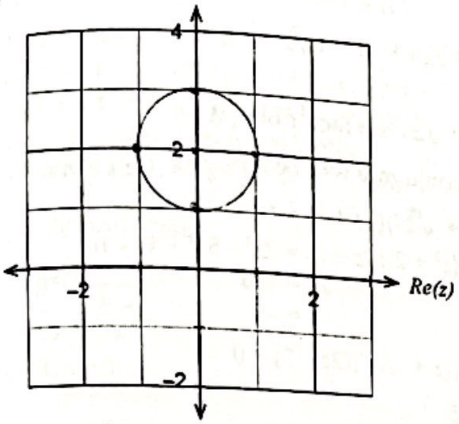
$k = 0$

Only solution that is the same for both is cis0.
So only 1 of these roots.

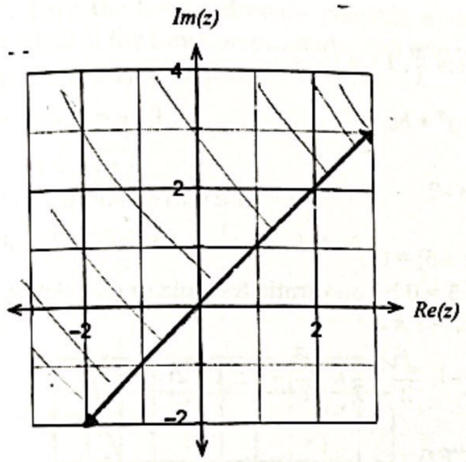
Chapter 3: Graphing in the Complex Plane

1. (MSPEC 2016:CA10)

(a)



(b) Distance from z to $(1, -1)$ is greater than or equal to the distance from z to $(-1, 1)$.



(c) $|z+2| = |z-(-2+0i)|$

Require max distance from $(-2, 0)$ to a point z on the circle. This is the distance from $(-2, 0)$ through the centre to a point on the circumference.

Max distance = $\sqrt{2^2+2^2}+1 = \sqrt{8}+1$ or $2\sqrt{2}+1$

2. (MSPEC 2017:CF6)

(a) $|z-(0+1i)| = 1$

$\therefore |z-i| = 1$

(b) $\theta = \tan^{-1}(2)$

Since ray $\Rightarrow y = 2x$ and equation of circle is $x^2 + (y-1)^2 = 1$

point of intersection is

$$x^2 + (2x-1)^2 = 1$$
$$x^2 + 4x^2 - 4x + 1 = 1$$
$$5x^2 - 4x = 0$$
$$x(5x-4) = 0$$

$x = 0$ or $x = \frac{4}{5}$

$x = \frac{4}{5} \Rightarrow y = 2\left(\frac{4}{5}\right) = \frac{8}{5}$

$$r = \sqrt{x^2+y^2}$$
$$= \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{8}{5}\right)^2}$$
$$= \sqrt{\frac{80}{25}}$$
$$= \frac{4\sqrt{5}}{5}$$

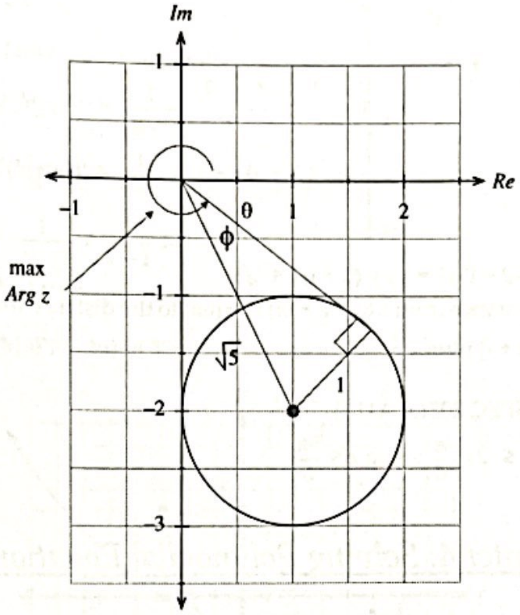
3. (MSPEC 2018:CA11)

(a) $|z-(1-2i)| \le 1$

(b) $\text{Arg}\,(z-(0+i)) = \frac{3\pi}{4}$

$\text{Arg}\,(z-i) = \frac{3\pi}{4}$

(c)



Distance from origin to centre is $\sqrt{1^2+2^2} = \sqrt{5}$

$\sin\phi = \frac{1}{\sqrt{5}}$ $\quad \phi = 0.46365$

$\tan(\theta+\phi) = \frac{2}{1}$ $\quad \theta+\phi = 1.10715$

$\therefore \theta = 0.6435$

Max Arg $z = 2\pi - 0.6435 = 5.64$

4. (MSPEC 2019:CA10)

(a) $\text{Arg}\,(z-(0-2i)) = \frac{\pi}{4}$ $\qquad z_0 = -2i,\ k = \frac{1}{4}$

(b) We require the minimum distance from $(0+i)$ to the half line above. i.e. the perpendicular distance from $(0, 1)$ to $(1.5, -0.5)$

Using a right triangle with these points and $(0, -0.5)$ minimum

distance = $\sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \frac{3\sqrt{2}}{2}$

5. (MSPEC 2020:CA10)

(a) The locus is the intersection of a circle and a region defined by half lines.

Circle is $|z-(i+2i)| = 2$

Half line region is $0 \le Arg\,(z-(i+2i)) \le \frac{2\pi}{3}$

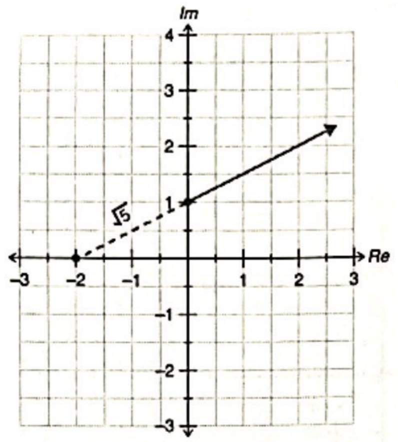
To get $\frac{2\pi}{3}$ we need the y intercept of the circle

$(x-1)^2 + (y-2)^2 = 4$ with $x = 0$ gives $(y-2)^2 = 3$

$y = 2+\sqrt{3}$ $(2-\sqrt{3}$ is lower).

Using right triangle with vertices $(1, 2)$, $(0, 2+\sqrt{3})$, $(0, 2)$ gives a $\frac{\pi}{3}$ angle in triangle so supplementary angle is $\frac{2\pi}{3}$.

(b)



$|z-(-2+0i)| = |z-(0+i)| + \sqrt{5}$

The distance from z to $(-2+0i)$ is equal to the distance from z to $(0+i)$ plus $\sqrt{5}$.

6. (MSPEC 2021:CA11d)

$|z-i| \leq 2 \cap \frac{\pi}{6} \leq \text{Arg}\, z \leq \frac{5\pi}{6}$

Chapter 4: Solving Polynomial Equations

1. (Projected CF)

(a) $2z-1$ is a factor $\therefore f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + a\left(\frac{1}{2}\right)^2 + b\left(\frac{1}{2}\right) - 1 = 0$

$z+1 \Rightarrow f(-1) = 2(-1)^3 + a(-1)^2 + b(-1) - 1 = -6$

Solve $\frac{1}{4}a + \frac{1}{2}b = \frac{3}{4}$

$a - b = -3 \qquad a = -1, b = 2$

(b) $f(z) = 2z^3 - z^2 + 2z - 1$

$$\begin{array}{r|l} & z^2+1 \\ \hline 2z-1 & 2z^3-z^2+2z-1 \end{array}$$

$\therefore f(z) = (2z-1)(z^2+1)$

$f(z) = 0 = (2z-1)(z+i)(z-i)$

$\therefore z = \frac{1}{2}, -i, i$

2. (MSPEC 2016:CF03)

(a) $f(-4) = (-4)^3 + 2(-4)^2 - 5(-4) + 12$

$= -64 + 32 + 20 + 12$

$= 0$

$\therefore z+4$ is a factor

(b) $(z+4)(z^2+bz+c) = 0$

$(z+4)(z^2+bz+3) = 0$

Equating z coefficient

$3z + 4bz = -5z$

$3 + 4b = -5$

$b = -2$

$(z+4)(z^2-2z+3) = 0$

$$z = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(3)}}{2}$$

$$= \frac{2 \pm \sqrt{-8}}{2}$$

$$= 1 \pm \sqrt{2}i$$

Solutions are $z = -4,\ 1 \pm \sqrt{2}i$

3. (MSPEC 2017:CF2)

(a) $f(\sqrt{2}\,i) = 2(\sqrt{2}\,i)^3 - 5(\sqrt{2}\,i)^2 + 4(\sqrt{2}\,i) - 10$

$= 2(-2\sqrt{2}\,i) - 5(-2) + 4\sqrt{2}\,i - 10$

$= -4\sqrt{2}\,i + 10 + 4\sqrt{2}\,i - 10$

$= 0$

$\therefore = z - \sqrt{2}\,i$ is a factor of $f(z)$

(b) $z + \sqrt{2}\,i$ (conjugate will be a factor)

(c) $(z - \sqrt{2}\,i)(z + \sqrt{2}\,i)(2z+k) = f(z)$

$(z^2+2)(2z+k) = 2z^3 - 5z^2 + 4z - 10$

$\therefore 2k = -10$

$k = -5$

$f(z) = (z - \sqrt{2}\,i)(z + \sqrt{2}\,i)(2z-5) = 0$

so $z = \pm\sqrt{2}\,i, 2.5$

4. (MSPEC 2018:CF7)

(a) $z^3 = -1$

$z = 1 \text{ cis } \pi, 1 \text{ cis } \frac{\pi}{3}, 1 \text{ cis } \left(\frac{-\pi}{3}\right)$

(b) $P(z) = (z^3+1)(z^2+bz+5)$

the z^4 term is

$-2z^4 = bz^4$ so $b = -2$

$Q(z) = z^2 - 2z + 5$

(c) $(z^3+1)(z^2-2z+5) = 0$

Solving $z^2 - 2z + 5 = 0$ by quadratic formula or completing the square gives $z = 1 \pm 2i$

Solutions are $z = -1,\ \frac{\sqrt{3}}{2} + \frac{1}{2}i,\ \frac{\sqrt{3}}{2} - \frac{1}{2}i,\ 1 \pm 2i$

5. (MSPEC 2019:CF2)

(a) $P(2i) = (2i)^4 - 2(2i)^3 + 14(2i)^2 - 8(2i) + 40$

$= 16 + 16i - 56 - 16i + 40$

$= 0$

$\therefore z - 2i$ is a factor

(b) $P(z) = (z-2i)(z+2i)(z^2+bz+c)$

$= (z^2+4)(z^2+bz+10)$

Equating co-efficients of z^3 term.

$bz^3 = -2z^3$

$\therefore b = -2$

$\therefore P(z) = (z^2+4)(z^2-2z+10)$

$z = \frac{2 \pm \sqrt{-36}}{2} = 1 \pm 3i$

All solutions are $1 \pm 3i, \pm 2i$

6. (MSPEC 2021:CF6)

(a) $P(z) = (z-(2+4i))(z-(2-4i))\,Q(z)$

$P(z) = (z^2 - 4z + 20)\,Q(z)$

So $z^2 - 4z + 20$ is a quadratic factor of $P(z)$

(b) $z^4 - 6z^3 + 31z^2 - 52z + 60 = (z^2 - 4z + 20)(z^2 + bz + 3)$

Equating co-efficients of z^3

$-6 = -4 + b$

$b = -2$

Solving $z^2 - 2z + 3 = 0$

$z = \frac{2 \pm \sqrt{4-12}}{2}$

$= 1 \pm \sqrt{2}\,i$

Full solution is $z = 2 \pm 4i, 1 \pm \sqrt{2}\,i$

Chapter 5: Composite and Inverse Functions

1. (Projected CF)

(a) $x \leq 1, y \geq 0$ (b) $f(g(x)) = x, g(f(x)) = |x|$

(c) $x \leq 1, y \leq 1$ (d) $g^{-1}(x) = 1 - x^2, x \geq 0$

(e) f and g are inverses of each other

(f) domain is $x \leq 1$, range $y \geq 2$

2. (MSPEC 2016:CF01)

(a) $g(f(x)) = \dfrac{1}{\ln x}$

(b) (i) $x > 0, x \neq 1$

(ii) $y \in \mathrm{R}, y \neq 0$

3. (MSPEC 2016:CF08c,d)

(a) $k = 1$

To have the largest domain possible must consider only left half of the function otherwise not one to one.

(b) $x = (y - 1)^2 - 4$

$\pm\sqrt{x+4} = y - 1$

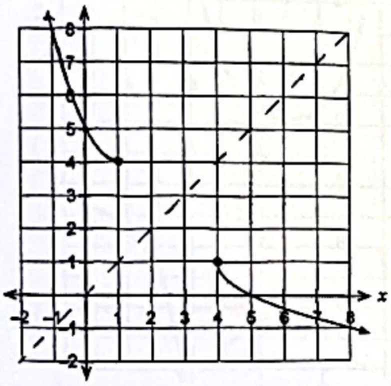
$y = 1 \pm \sqrt{x+4}$

to have a range of $y \leq 1$

Require $f^{-1}(x) = 1 - \sqrt{x+4}, x \geq -4$

4. (MSPEC 2017:CF4)

(a)



(b) $\sqrt{y-4} = 1 - x$

$y - 4 = (1 - x)^2$

$y = f^{-1}(x) = (1 - x)^2 + 4$

Since R_f is $\{y \mid y \leq 1\}$

$\therefore \mathrm{D}_{f\text{-}1}$ is $\{x \mid x \leq 1\}$

(c) $f \circ g(x) = f\left(\dfrac{1}{x^2}\right) = 1 - \sqrt{\dfrac{1}{x^2} - 4}$

(d) Using the unsimplified rule for $f \circ g(x)$

$$\frac{1}{x^2} \geq 4$$

$$x^2 \leq \frac{1}{4}$$

solution $-\dfrac{1}{2} \leq x \leq \dfrac{1}{2}$

So $\mathrm{D}_{f \circ g}$ is $\left\{x \mid -\dfrac{1}{2} \leq x \leq \dfrac{1}{2},\ x \neq 0\right\}$

5. (MSPEC 2018:CF1)

(a) $g \circ f(x) = \dfrac{\sqrt{x-3}}{\sqrt{x-3} - 2}$

(b) $x \geq 3, x \neq 7$

(c) Not true. Since $D_{f\text{-}1}$ is $x \geq 0$ (R_f is $y \geq 0$)

$x = -1$ is not within $D_{f\text{-}1}$

6. (MSPEC 2019:CF4)

(a) $\dfrac{1}{\sqrt{x} - 1}$ (b) $x \geq 0, x \neq 1$ (c) $y \leq -1, y > 0$

(d) Now $\sqrt{x^2} = |x|$

So $f(h(g(x))) = \dfrac{1}{|x| - 1}$

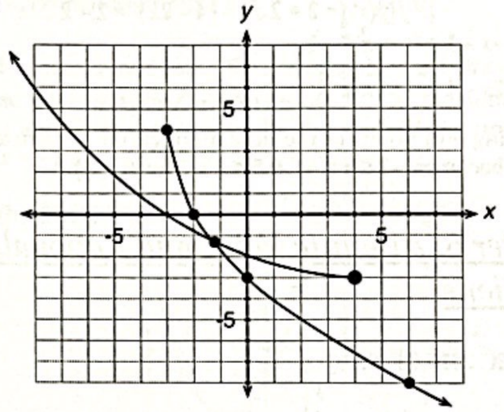
$\therefore f(h(g(x))) = \dfrac{1}{x - 1}$ for $x \geq 0, x \neq 1$ only

But $\dfrac{1}{|x| - 1} \neq \dfrac{1}{x - 1}$ for $x < 0$

$\therefore$ Not true

7. (MSPEC 2019:CF5)

(a)



(b) $x = \dfrac{1}{16}(y - 4)^2 - 3$

$\pm\sqrt{16(x+3)} = y - 4$

$4 \pm \sqrt{16(x+3)} = y$

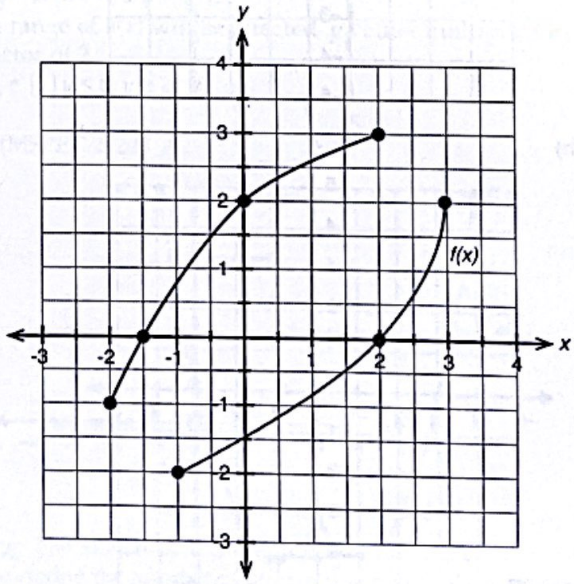
Since $D_{g(x)}$ is $x \leq 4$

$\therefore Rg^{-1}(x)$ is $y \leq 4$

and $g^{-1}(x) = 4 - \sqrt{16(x+3)}, x \geq -3$

8. (MSPEC 2021:CF2)

(a)



(b) g is not a one-to-one function. It fails the horizontal line test.

(c) $x = 2 - 2\sqrt{3-y}$

$2\sqrt{3-y} = 2 - x$

$3 - y = \left(\frac{2-x}{2}\right)^2$

$f^{-1}(x) = 2 + x - \frac{x^2}{4}, -2 \le x \le 2$

(d) $f(0) = 2 - 2\sqrt{3}$

$g(f(0)) = (2 - 2\sqrt{3}) + 4$ where $g(x) = x + 4, -2 \le x \le 0$

$= 6 - 2\sqrt{3}$

(e) Since the range of $g(x)$ is $0 \le y \le 4$ and the domain of $f(x)$ is $-1 \le x \le 3$ these must both be restricted to $0 \le y \le 3$ and $0 \le x \le 3$ respectively (i.e. overlapping). Working backwards with the restricted range of $g(x)$ gives a restricted domain of $-2 \le x \le -1, 0.5 \le x \le 2$ for $f(g(x))$.

Alternatively use $f(g(x)) = 2 - 2\sqrt{3-(x+4)} = 2 - 2\sqrt{-x+1}$

or $f(g(x)) = 2 - 2\sqrt{3-(4-2x)} = 2 - 2\sqrt{2x-1}$

Solving $-x - 1 \ge 0 \Rightarrow x \le -1$

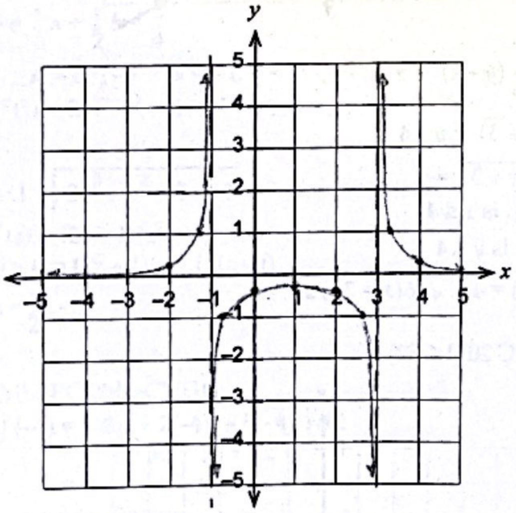
$2x - 1 \ge 0 \Rightarrow x \ge \frac{1}{2}$

Then using end points of the domain of $g(x)$ the restricted domain becomes $-2 \le x \le -1, 0.5 \le x \le 2$ for $f(g(x))$.

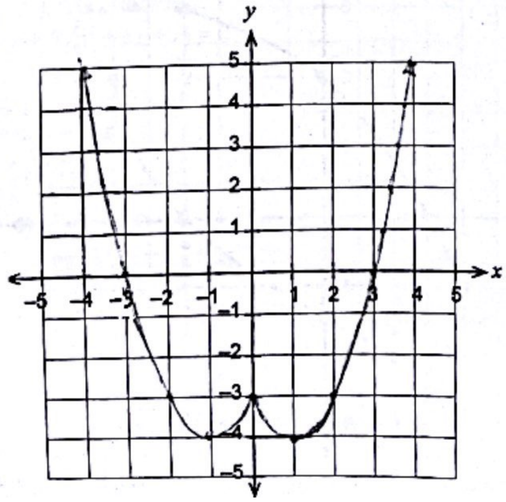
Chapter 6: Absolute Value and Rational Functions

1. (MSPEC 2016:CF08a,b)

(a)



(b)



2. (MSPEC 2016:CA12)

(a) $a = -2, b = 3, c = 5$

(b) Horizontal line given by $y = d$ would cut the graph that looks like ~~W~~ at 4 points for y values given by $0 < d < 5$.

3. (MSPEC 2017:CF5)

$f(x) = \frac{-4(x-3)(x+1)}{(x-4)(x+2)}$

vertical asy. $x = -2, x = 4$

horizontal asy. $x \to \pm\infty$

$y \to \frac{-4x^2}{x^2}$

$\therefore y = -4$

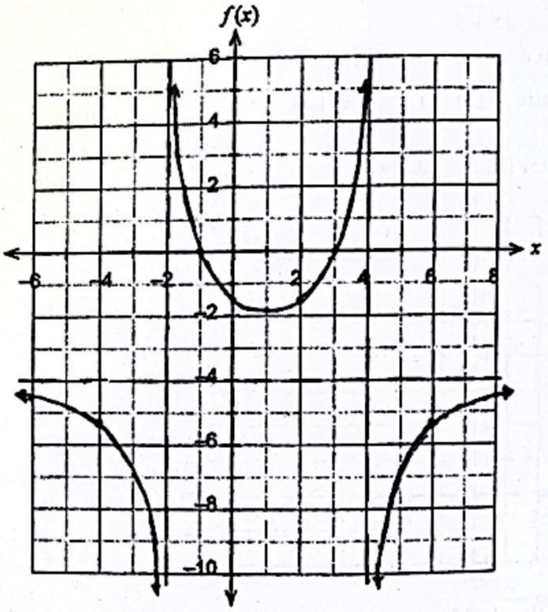
x intercepts $x = 3, -1$

y intercepts $y = \frac{-4(-3)(1)}{(-4)(2)}$

$y = -1.5$

Other points

x	-4	1	6
y	$\frac{-21}{4}$	$\frac{-16}{9}$	$\frac{-21}{4}$



4. (MSPEC 2017:CA16a)

Gradient of arms $= \pm\frac{b}{a}$

$\therefore f(x) = \frac{b}{a}|x - a|$

5. (MSPEC 2018:CF4)

vertical asymptotes at $x = c, x = d$

$\therefore c = -2, d = 3$

x intercepts at $x = a, x = b$ $(f(x) = 0)$

$\therefore a = -3, b = 1$

horizontal asymptote at $y = 2$.

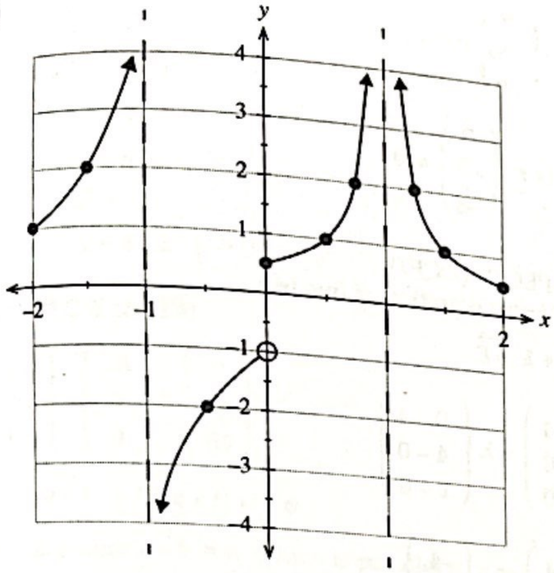
Dominant powers has

$f(x) = \frac{kx^2}{x^2}$

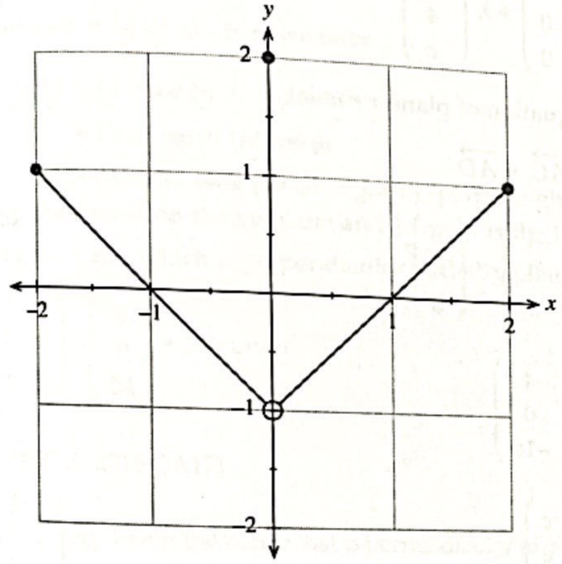
$\therefore k = 2.$

6. (MSPEC 2018:CA14)

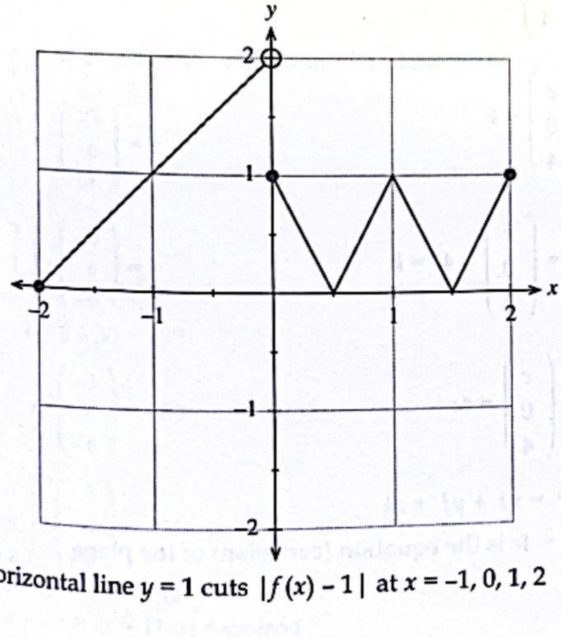
(a)



(b)

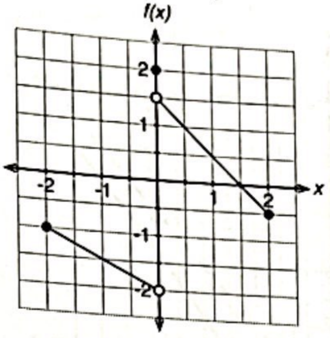


(c)



Horizontal line $y = 1$ cuts $|f(x) - 1|$ at $x = -1, 0, 1, 2$

7. (MSPEC 2019:CF7)



$f(x)$, drawn like this, passes both the horizontal and vertical line tests since $f^{-1}(x)$ is also a function.

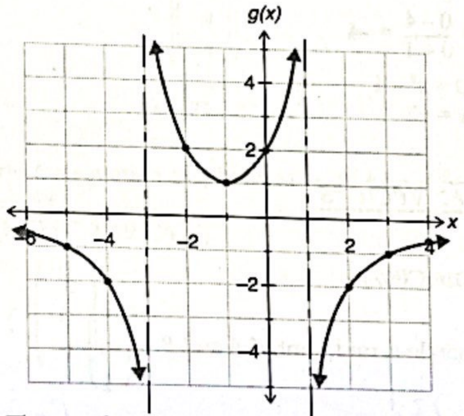
8. (MSPEC 2020:CF3)

a	b	c	d
2	4	1	3

- a is determined using dominant powers to give the horizontal asymptote of $y = 2$ $\left(\frac{ax^2}{x^2} = y\right)$.
- b is determined using the x intercepts which are –2 and 2 so $x^2 - b$ must be $x^2 - 4 = (x - 2)(x + 2)$ on the numerator.
- c and d are determined by the vertical asymptotes at $x = -1$ and $x = 3$.

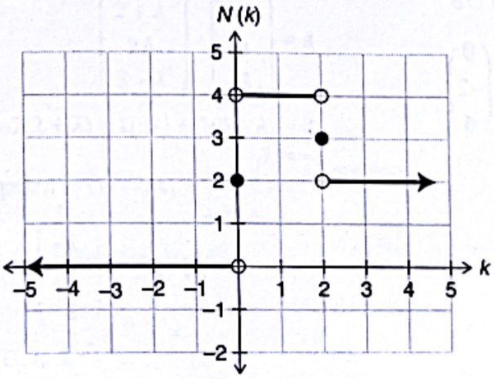
9. (MSPEC 2020:CF5)

(a)



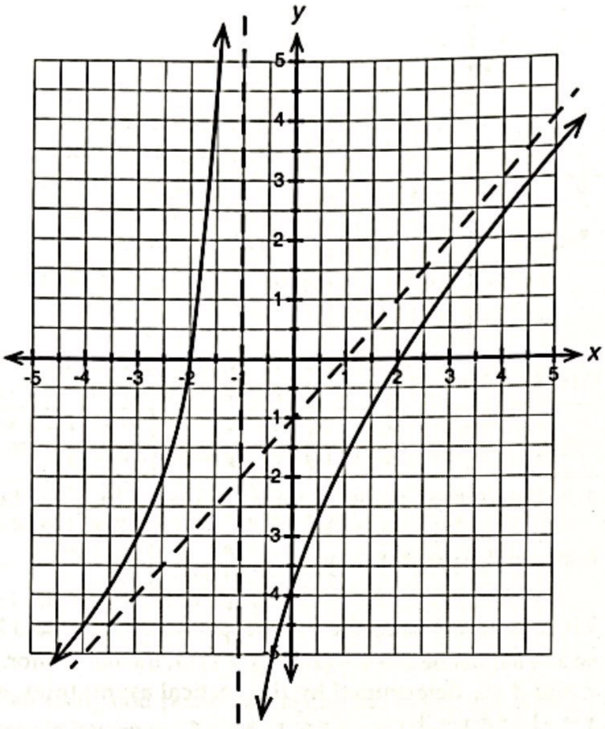
(b) The graph of $h(x)$ will be vertical dilation of factor 2 of the graph in (a).
Hence x values not affected by vertical dilation.
$D_{h(x)} = \{x \mid x \varepsilon R, x \neq -3, 1\}$.
The range of $h(x)$ will be affected. y values multiplied by a factor of 2.
$R_{h(x)} = \{y \mid y < 0, y \geq 2\}$.

10. (MSPEC 2020:CA21)



Note: The sketch of the graph above of $N(k)$ is done by considering the number of intersection points of the graphs of $y = |f(x)|$ and $y = k$ (horizontal lines, $k \varepsilon R$).

11. (MSPEC 2021:CF4)



Oblique asymptote at $y = x - 1$
Vertical asymptote at $x = -1$

y intercept $= \dfrac{0-4}{0+1} = -4$

x intercepts $0 = x^2 - 4$
$x = \pm 2$

Chapter 7: Vectors

1. (MSPEC 2016:CF07)

(a) Centre of circle is midpoint of A and $B = \begin{pmatrix}2\\-1\\4\end{pmatrix}$

$$\text{radius} = \left|\begin{pmatrix}4\\0\\3\end{pmatrix} - \begin{pmatrix}2\\-1\\4\end{pmatrix}\right|$$

$$= \sqrt{6}$$

$\therefore$ Vector Equation of circle is $\left|\underset{\sim}{r} - \begin{pmatrix}2\\-1\\4\end{pmatrix}\right| = \sqrt{6}$

(b) $\underset{\sim}{n} = \overrightarrow{OA} \times \overrightarrow{OB}$

$$= \begin{pmatrix}4\\0\\3\end{pmatrix} \times \begin{pmatrix}0\\-2\\5\end{pmatrix}$$

$$= \begin{pmatrix}6\\-20\\-8\end{pmatrix}$$

$$\underset{\sim}{r} \cdot \begin{pmatrix}6\\-20\\-8\end{pmatrix} = c$$

$$\begin{pmatrix}0\\0\\0\end{pmatrix} \cdot \begin{pmatrix}6\\-20\\-8\end{pmatrix} = 0 = c$$

plane is $\underset{\sim}{r} \cdot \begin{pmatrix}6\\-20\\-8\end{pmatrix} = 0$

2. (MSPEC 2017:CF7)

(a) (i) Vector equation of line use

$$\underset{\sim}{r} = A + \lambda\,\overrightarrow{AE}$$

$$= \begin{pmatrix}4\\0\\0\end{pmatrix} + \lambda \begin{pmatrix}0-4\\4-0\\c-0\end{pmatrix}$$

$$= \begin{pmatrix}4\\0\\0\end{pmatrix} + \lambda \begin{pmatrix}-4\\4\\c\end{pmatrix}$$

(ii) Equation of plane $\underset{\sim}{r} \bullet \underset{\sim}{n} = k$

$$\underset{\sim}{n} \Rightarrow \overrightarrow{AC} \times \overrightarrow{AD}$$

$$\Rightarrow \begin{pmatrix}-4\\0\\c\end{pmatrix} \times \begin{pmatrix}0\\4\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}-4c\\0\\-16\end{pmatrix}$$

$$\underset{\sim}{n} = \begin{pmatrix}c\\0\\4\end{pmatrix}$$

$$\underset{\sim}{r} \bullet \begin{pmatrix}c\\0\\4\end{pmatrix} = k$$

$$\begin{pmatrix}4\\4\\0\end{pmatrix} \bullet \begin{pmatrix}c\\0\\4\end{pmatrix} = 4c = k$$

$$\therefore \underset{\sim}{r} \bullet \begin{pmatrix}c\\0\\4\end{pmatrix} = 4c$$

Since $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$cx + 4z = 4c$ is the equation (cartesian) of the plane.

(b) Require $\overrightarrow{AE} \bullet \overrightarrow{BG} = 0$

$$\begin{pmatrix}-4\\4\\c\end{pmatrix}\bullet\begin{pmatrix}4\\-4\\c\end{pmatrix}=0$$

$$-16-16+c^2=0$$
$$c^2=32$$
$$c=4\sqrt{2}\quad (c>0)$$

3. (MSPEC 2018:CF8)

(a) $\begin{pmatrix}3\\6\\0\end{pmatrix}\times\begin{pmatrix}-8\\2\\0\end{pmatrix}=\begin{pmatrix}0\\0\\54\end{pmatrix}$

(b) $\underset{\sim}{c}\cdot(\underset{\sim}{a}\times\underset{\sim}{b}) = |\underset{\sim}{c}|\,|\underset{\sim}{a}\times\underset{\sim}{b}|\cos\phi$

angle is ϕ since $\underset{\sim}{a}\times\underset{\sim}{b} = \underset{\sim}{n}$ which is parallel to z axis.

Since $\underset{\sim}{a}\times\underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}|\,\sin\theta\;\hat{\underset{\sim}{n}}$ we have

$$\underset{\sim}{c}\cdot(\underset{\sim}{a}\times\underset{\sim}{b}) = |\underset{\sim}{c}|\cos\phi\,|\underset{\sim}{a}|\,|\underset{\sim}{b}|\,|\hat{\underset{\sim}{n}}|\,\sin\theta$$
$$= (|\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta)\,(|\underset{\sim}{c}|\cos\phi)$$

= (Area of base parallelogram) (perp. height)

<u>Note</u>: the base is on the xy plane and $|\underset{\sim}{c}|\cos\phi$ is the height along the z axis which is perpendicular to the xy plane.

(c) $\begin{pmatrix}-1\\2\\5\end{pmatrix}\cdot\begin{pmatrix}0\\0\\54\end{pmatrix}=270$ units^3

4. (MSPEC 2018:CA17)

(a) $\begin{pmatrix}-3\\6\\-6\end{pmatrix}$ represents the vector that is perpendicular to plane Π

(b) $\underset{\sim}{r}\cdot\underset{\sim}{n} = c$ is also the equation of a plane.

$$\underset{\sim}{r}\cdot\begin{pmatrix}-3\\6\\-6\end{pmatrix}=c$$

$$\begin{pmatrix}3\\1\\5\end{pmatrix}\cdot\begin{pmatrix}-3\\6\\-6\end{pmatrix}=c$$

$$c=-9+6-30=-33$$

$$\underset{\sim}{r}\cdot\begin{pmatrix}-3\\6\\-6\end{pmatrix}=-33$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}-3\\6\\-6\end{pmatrix}=-33$$

$$-3x+6y-6z=-33$$

$x-2y+2z=11$ as required

(c) $\underset{\sim}{r}=\begin{pmatrix}3\\4\\-1\end{pmatrix}+\lambda\begin{pmatrix}-3\\6\\-6\end{pmatrix}$

is the equation of the line that passes through the centre of the sphere along a radius to the point of contact on the sphere and the plane, $\begin{pmatrix}-3\\6\\-6\end{pmatrix}$ is perpendicular to the plane as the radius and the plane are perpendicular at the point of contact.

Substitute vector equation of line above into

$$\underset{\sim}{r}\cdot\begin{pmatrix}-3\\6\\-6\end{pmatrix}=-33$$

$$\begin{pmatrix}3-3\lambda\\4+6\lambda\\-1-6\lambda\end{pmatrix}\cdot\begin{pmatrix}-3\\6\\-6\end{pmatrix}=-33$$

$$-3(3-3\lambda)+6(4+6\lambda)-6(-1-6\lambda)=-33$$
$$\lambda=\frac{-2}{3}$$

point of contact at $\underset{\sim}{r}=\begin{pmatrix}5\\0\\3\end{pmatrix}$

$$\text{Length of radius} = \left|\begin{pmatrix}5\\0\\3\end{pmatrix}-\begin{pmatrix}3\\4\\-1\end{pmatrix}\right|$$
$$=\left|\begin{pmatrix}2\\-4\\4\end{pmatrix}\right|$$
$$=\sqrt{2^2+4^2+4^2}$$
$$=6$$

Cartesian equation of sphere is $(x-3)^2+(y-4)^2+(z+1)^2=36$

5. (MSPEC 2019:CA16)

(a) $\underset{\sim}{r}\cdot\begin{pmatrix}-2\\-1\\1\end{pmatrix}=4$

So $\begin{pmatrix}-2\\-1\\1\end{pmatrix}$ is one such vector

(b) Substitute $\begin{pmatrix}2+\lambda\\2\lambda\\3-\lambda\end{pmatrix}$ into plane equation

$$\begin{pmatrix}2+\lambda\\2\lambda\\3-\lambda\end{pmatrix}\cdot\begin{pmatrix}-2\\-1\\1\end{pmatrix}=4$$

$$-2(2+\lambda)-1(2\lambda)+1(3-\lambda)=4$$
$$\lambda=-1$$

point is $(1,-2,4)$

(c) $\begin{pmatrix}-2\\-1\\1\end{pmatrix}$ is ⊥ to Π_1

Π_1 is ⊥ to Π_2

$\therefore \begin{pmatrix}-2\\-1\\1\end{pmatrix}$ is parallel to or can lie on Π_2

Since L_1 is on Π_2 then $\begin{pmatrix}1\\2\\-1\end{pmatrix}$ lies on Π_2

$$\underset{\sim}{n} = \begin{pmatrix}1\\2\\-1\end{pmatrix} \times \begin{pmatrix}-2\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\3\end{pmatrix}$$

$$\underset{\sim}{r} \cdot \begin{pmatrix}1\\1\\3\end{pmatrix} = c$$

$$\begin{pmatrix}2\\0\\3\end{pmatrix} \cdot \begin{pmatrix}1\\1\\3\end{pmatrix} = 11$$

$$\therefore \underset{\sim}{r} \cdot \begin{pmatrix}1\\1\\3\end{pmatrix} = 11$$

(d) Substitute $\begin{pmatrix}2+\lambda\\2\lambda\\3-\lambda\end{pmatrix}$ into equation of sphere

$$\left|\begin{pmatrix}2+\lambda\\2\lambda\\3-\lambda\end{pmatrix} - \begin{pmatrix}3\\1\\4\end{pmatrix}\right| = \sqrt{35}$$

Solve $(\lambda-1)^2 + (2\lambda-1)^2 + (-\lambda-1)^2 = 35$
$\lambda = -2, 2\frac{2}{3}$
2 solutions so 2 points of intersection
$\therefore$ not tangential

6. (MSPEC 2019:CA19)
Let the equation of the line emanating from (2, 3, -7) on and perpendicular to plane Π_1 be given by

$$\underset{\sim}{r} = \begin{pmatrix}2\\3\\-7\end{pmatrix} + \lambda\begin{pmatrix}a\\b\\c\end{pmatrix}$$

This line intersects plane Π_2 at some point ($\lambda = \lambda_1$)

$$\text{So} \begin{pmatrix}2+\lambda_1 a\\3+\lambda_1 b\\-7+\lambda_1 c\end{pmatrix} \cdot \begin{pmatrix}a\\b\\c\end{pmatrix} = -4$$

$a(2+\lambda_1 a) + b(3+\lambda_1 b) + c(-7+\lambda_1 c) = -4$
$\lambda_1(a^2+b^2+c^2) = -4 - (2a+3b-7c)$ — Eq 1

Π_1 has equation $\underset{\sim}{r} \cdot \begin{pmatrix}a\\b\\c\end{pmatrix} = 11$

(2, 3, -7) is on Π_1 so $2a + 3b - 7c = 11$
Substitute into Eq 1
$\lambda_1(a^2+b^2+c^2) = -4 - 11 = -15$

$$\lambda_1 = \frac{-15}{a^2+b^2+c^2}$$

The distance between the planes is the distance from (2, 3, -7) on Π_1 to $\underset{\sim}{r} = \begin{pmatrix}2\\3\\7\end{pmatrix} + \frac{-15}{a^2+b^2+c^2}\begin{pmatrix}a\\b\\c\end{pmatrix}$ on Π_2 which is given by

$$\left|\begin{pmatrix}2\\3\\-7\end{pmatrix} - \frac{15}{a^2+b^2+c^2}\begin{pmatrix}a\\b\\c\end{pmatrix} - \begin{pmatrix}2\\3\\-7\end{pmatrix}\right| = \left|\frac{-15}{a^2+b^2+c^2}\begin{pmatrix}a\\b\\c\end{pmatrix}\right|$$

$$= \frac{15}{a^2+b^2+c^2}\sqrt{a^2+b^2+c^2}$$

$$= \frac{15}{\sqrt{a^2+b^2+c^2}} \text{ as required}$$

7. (MSPEC 2020:CF2)

(a) $\underset{\sim}{n} = \begin{pmatrix}3\\0\\1\end{pmatrix} \times \begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}1\\-5\\-3\end{pmatrix}$

(b) $\underset{\sim}{r} \cdot \begin{pmatrix}1\\-5\\-3\end{pmatrix} = c$

$$\begin{pmatrix}0\\0\\4\end{pmatrix} \cdot \begin{pmatrix}1\\-5\\-3\end{pmatrix} = c$$

$$c = -12 \therefore \underset{\sim}{r} \cdot \begin{pmatrix}1\\-5\\-3\end{pmatrix} = -12$$

Cartesian equation is $x - 5y - 3z = -12$

8. (MSPEC 2020:CA20a)
$\overrightarrow{PQ} = (1-k)\underset{\sim}{a} + k\underset{\sim}{b}$

$\overrightarrow{QR} = (1-k)\underset{\sim}{b} - k\underset{\sim}{a}$

Using the dot product of these vectors
$[(1-k)\underset{\sim}{a} + k\underset{\sim}{b}] \cdot [-k\underset{\sim}{a} + (1-k)\underset{\sim}{b}]$
$= -(1-k)k\,|\underset{\sim}{a}|^2 + (1-k)k\,|\underset{\sim}{b}|^2$
Since square $|\underset{\sim}{a}| = |\underset{\sim}{b}|$
$= 0 \therefore \angle PQR = 90°$

9. (MSPEC 2021:CA16)
(a) B is (2, 4, 0)

$\overrightarrow{BT}$ is $\begin{pmatrix}-2\\-4\\3\end{pmatrix}$

$$\underset{\sim}{r} = \begin{pmatrix}2\\4\\0\end{pmatrix} + \lambda\begin{pmatrix}-2\\-4\\3\end{pmatrix}, \ \lambda \varepsilon [0, 1]$$

(b) Centre of sphere must be centre of prism.
Centre is (1, 2, 1.5)
radius → distance from centre to (0, 0, 0).

$$\text{radius} = \sqrt{1^2 + 2^2 + 1.5^2} = \frac{\sqrt{29}}{2}$$

Equation of sphere is $(x-1)^2 + (y-2)^2 + (z-1.5)^2 = \frac{29}{4}$

(c) Vector equation of $\overleftrightarrow{AM}$ is

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 2 \\ 1.5 \end{pmatrix}$$

Paths cross when

$2 - 2\mu = 2 - 2\lambda$
$2\mu = 4 - 4\lambda$
$1.5\mu = 3\lambda$
$\mu = 2\lambda$

Substituting into equation 2 gives

$2(2\lambda) = 4 - 4\lambda$
$8\lambda = 4$
$\lambda = 0.5$

Substituting into equation 1 gives

$2 - 2(2\lambda) = 2 - 2\lambda$
$\lambda = 0$

Since linear paths, there can be only 1 point of intersection.
So contradiction that $\lambda = 0$ and $\lambda = 0.5$.
No solution ∴ paths do not cross.

Chapter 8: Applications of Vectors in Three Dimensions

1. (Projected CA)
Solve simultaneous equations on Classpad

$10 + 3t = 28 + 2s$
$-5 + t = 22 - 4s$
$5 - 2t = -31 + 4s$

Solution is $t = 9 \quad s = 4.5$
So no collision but paths do cross at 37i + 4j − 13k

2. (3CDMAS 2015:CA18)

(a) $\text{Sp} = \sqrt{10^2 + 10^2 + 5^2}$
$= 15\ \text{km/hr}$

(b) $\underset{\sim}{r} = [22, -23, -5]$ km

(c) $\underset{\sim}{r} = \begin{pmatrix} -20 \\ 10 \\ -30 \end{pmatrix} + t \begin{pmatrix} 30 \\ -30 \\ -15 \end{pmatrix}$

(d) $D = |\underset{\sim}{r}_B - \underset{\sim}{r}_A|$

$$= \left| \begin{pmatrix} 42 - 20t \\ 20t - 33 \\ 10t + 25 \end{pmatrix} \right|$$

$$D = \sqrt{(42-20t)^2 + (20t-33)^2 + (10t+25)^2}$$

(e) $D = \sqrt{900x^2 - 2500x + 3478}$

$$\frac{dD}{dt} = \frac{1800x - 2500}{\sqrt{900x^2 - 2500x + 3478}} \cdot \frac{1}{2}$$

Max or Min at $1800x - 2500 = 0$

$x = 1.3\dot{8}$ hours
$D = 41.74$ km

(f) 5.23 pm

3. (MSPEC 2016:CA20)

(a) Vector Equation of line $\overrightarrow{SB}$ is

$$\underset{\sim}{r} = \begin{pmatrix} -2 \\ 3 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

point of intersection of line and plane

$$\begin{pmatrix} -2 + \lambda \\ 3 + \lambda \\ 6 - \lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 9$$

$\lambda = 6$

So $B = \begin{pmatrix} 4 \\ 9 \\ 0 \end{pmatrix}$

(b) Let $\hat{\underset{\sim}{d}}_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$-\underset{\sim}{d}_1 \cdot \underset{\sim}{n} = \sqrt{1^2 + 1^2 + 1^2}\ \sqrt{1^2 + 2^2}\ \cos\theta = 1$$

$$\cos\theta = \frac{1}{\sqrt{15}}$$

$$\hat{\underset{\sim}{d}}_2 \cdot \underset{\sim}{n} = \sqrt{1} \cdot \sqrt{1^2 + 2^2}\ \cos\theta = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$b + 2c = \frac{1}{\sqrt{3}} \text{ — Equation 1}$$

$$-\underset{\sim}{d}_1 \cdot \hat{\underset{\sim}{d}}_2 = \sqrt{3}\ \sqrt{1}\ \cos 2\theta$$

$$\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \sqrt{3}\ \sqrt{1}\ \left(\frac{-13}{15}\right)$$

$$-a - b + c = \frac{-13\sqrt{3}}{15} \text{ — Equation 2}$$

$$-\underset{\sim}{d}_1 \times \underset{\sim}{n} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$$

$$\hat{\underset{\sim}{d}}_2 \cdot \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} = -3a + 2b - c = 0 \text{ — Equation 3}$$

Solving 3 equations on Classpad gives

$\hat{\underset{\sim}{d}} = 0.58\hat{\underset{\sim}{i}} + 0.81\hat{\underset{\sim}{j}} - 0.12\hat{\underset{\sim}{k}}$ (2 d.p.)

4. (MSPEC 2017:CA14)

(a) $\underset{\sim}{r}(120) = \begin{pmatrix} 160 \\ 72 \\ 47.6 \end{pmatrix}$

∴ 47.6 m high

(b) $D\underset{\sim}{r}_T$ or $\overrightarrow{TD} = \underset{\sim}{r}_D - \underset{\sim}{r}_T$

$$= \begin{pmatrix} 100 + 0.5t \\ 0.6t \\ 50 - 0.02t \end{pmatrix} - \begin{pmatrix} 200 \\ 150 \\ 30 \end{pmatrix}$$

$$= \begin{pmatrix} 0.5t - 100 \\ 0.6t - 150 \\ 20 - 0.02t \end{pmatrix}$$

(c) Interference when

$$\left|D\underset{\sim}{r}_T\right| \text{ or } \left|\overrightarrow{TD}\right| \leq 50$$

$$\left|\begin{pmatrix} 0.5t-100 \\ 0.6t-150 \\ 20-0.02t \end{pmatrix}\right| \leq 50$$

Solve on Classpad

$$\sqrt{(0.5t-100)^2+(0.6t-150)^2+(20-0.02t)^2} \leq 50$$

$174.31 \leq t \leq 285.71$

$285.71 - 174.31 = 111.4$

$\therefore$ 111 seconds

Chapter 9: Vector Calculus in Two Dimensions

1. (MSPEC 2016:CA16)

(a) $x = 4\cos 2t \qquad y = 2\cos t$

$x = 4(2\cos^2 t - 1)$

$x = 2(2\cos t)^2 - 4$

$x = 2y^2 - 4, -4 \leq x \leq 4$

(b) $S = |\mathbf{v}(t)| = \left|\begin{pmatrix} -8\sin 2t \\ -2\sin t \end{pmatrix}\right|$

$$S = \sqrt{(-8\sin(2t))^2 + (-2\sin t)^2}$$

at $x = -2 \qquad -2 = 4\cos 2t$

$t = 1.047 \text{ sec} \cdot \frac{\pi}{3}$

$S = 7.14$ cm/sec (2 d.p.)

(c) $(4\cos 0, 2\cos 0) = (4, 2)$ so starts at A $(t = 0)$

Since $\cos\pi = -1$ so $t = \pi$ ends at B $(4, -2)$

$$\text{Distance} = \int_0^{\pi} \sqrt{(-8\sin(2t))^2 + (-2\sin t)^2}\, dt$$

2. (MSPEC 2017:CA15)

(a) $\underset{\sim}{v}(0) = 2\underset{\sim}{j}$

On the diagram this vector should start at $A(3,0)$ and go vertically up 2 units with arrowhead at $(3,2)$

(b) $\displaystyle\int_0^{\frac{3\pi}{2}} -\sin\left(\frac{t}{3}\right)\underset{\sim}{i} + 2\cos t\underset{\sim}{j}\, dt$

(c) $\underset{\sim}{r}(t) = \displaystyle\int -\sin\left(\frac{t}{3}\right)\underset{\sim}{i} + 2\cos t\underset{\sim}{j}\, dt$

$$= 3\cos\left(\frac{t}{3}\right)\underset{\sim}{i} + 2\sin t\underset{\sim}{j} + \underset{\sim}{c}$$

$\underset{\sim}{r}(0) = 3\underset{\sim}{i} = 3\cos 0\underset{\sim}{i} + 2\sin 0\underset{\sim}{j} + \underset{\sim}{c}$

$\underset{\sim}{c} = 0\underset{\sim}{i} + 0\underset{\sim}{j}$

$\therefore \underset{\sim}{r}(t) = \begin{pmatrix} 3\cos\frac{t}{3} \\ 2\sin t \end{pmatrix}$

(d) $\underset{\sim}{a}(t) = \begin{pmatrix} -\frac{1}{3}\cos\frac{t}{3} \\ -2\sin t \end{pmatrix}$

Velocity vector horizontal at point B, i.e. $\underset{\sim}{v}_B = k\underset{\sim}{i} + 0\underset{\sim}{j}$

Since $\underset{\sim}{j}$ component of velocity is $2\cos t$ then

$2\cos t = 0$

$t = \frac{\pi}{2}$ (the first time)

$\underset{\sim}{a}\left(\frac{\pi}{2}\right) = -0.29\underset{\sim}{i} - 2\underset{\sim}{j}$

(e) Time for 1 lap is

$T = \dfrac{2\pi}{\frac{1}{3}} = 6\pi$ seconds

$$\text{Distance} = \int_0^{6\pi} \left|\underset{\sim}{v}(t)\right| dt$$

$$= \int_0^{6\pi} \sqrt{\left(-\sin\frac{t}{3}\right)^2 + (2\cos t)^2}\, dt$$

$= 28.16$ m

3. (MSPEC 2018:CA19)

(a) $\underset{\sim}{v}(t) = \int -9.8\underset{\sim}{j}\, dt + w\underset{\sim}{i}$

$\underset{\sim}{v}(t) = -9.8t\underset{\sim}{j} + \underset{\sim}{c}_1 + w\underset{\sim}{i}$

$\underset{\sim}{v}(0) = 70\cos\theta\underset{\sim}{i} + 70\sin\theta\underset{\sim}{j} = \underset{\sim}{c}_1 \quad (t = 0, w = 0)$

$\therefore \underset{\sim}{v}(t) = (70\cos\theta + w)\, i + (70\sin\theta - 9.8t)\, \underset{\sim}{j}$

$\underset{\sim}{r}(t) = (70\cos\theta + w)\, t\underset{\sim}{i} + ((70\sin\theta)t - 4.9t^2)\, \underset{\sim}{j} + \underset{\sim}{c}_2$

$\underset{\sim}{r}(0) = 0\underset{\sim}{i} + 0\underset{\sim}{j} = \underset{\sim}{c}_2$

$\therefore \underset{\sim}{r}(t) = \begin{pmatrix} (70\cos\theta + w)t \\ (70\sin\theta)t - 4.9t^2 \end{pmatrix}$ as required

(b) $x = (70\cos\theta + w)\, t$

$y = (70\sin\theta)\, t - 4.9t^2$

$t = \dfrac{x}{70\cos\theta + w}$

Substitute into second equation

$$y = \frac{(70\sin\theta)x}{70\cos\theta + w} - \frac{4.9x^2}{(70\cos\theta + w)^2}$$

(c) range is given by the x intercept of the parabola in (b) (i.e. at $y = 0$)

Solve on Classpad

$$0 = \frac{(70\sin\theta)x}{70\cos\theta + 2} - \frac{4.9x^2}{(70\cos\theta + 2)^2}$$

$$0 = x\left(\frac{70\sin\theta}{70\cos\theta + 2} - \frac{4.9x}{(70\cos\theta + 2)^2}\right)$$

$$x = 0 \text{ or } x = \frac{70\sin\theta(70\cos\theta + 2)^2}{(70\cos\theta + 2)4.9}$$

$$x = \frac{70\sin\theta(70\cos\theta + 2)}{4.9}$$

Maximum x at $\frac{dx}{d\theta} = 0$ for $(0° < \theta < 90°)$

From Classpad $x = 45.6°$ $(45.573°)$

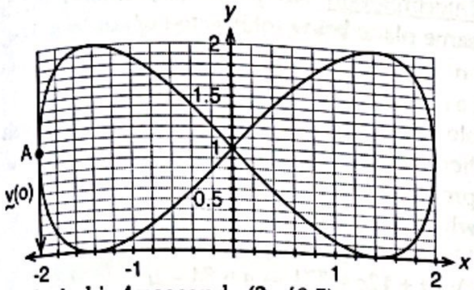
4. (MSPEC

(a) $t = 0 \quad \underset{\sim}{r}(o) = -2\underset{\sim}{i} + \underset{\sim}{j}$

so (−2, 1)

(b) $\underset{\sim}{v}(t) = \sin\left(\frac{t}{2}\right)\underset{\sim}{i} - \cos t\ \underset{\sim}{j}$

$t = 0 \quad \underset{\sim}{v} = -\underset{\sim}{j}$



(c) Period is 4π seconds $(2\pi/0.5)$

$$\text{Distance travelled} = \int_0^{4\pi} \left| \begin{pmatrix} \sin 0.5t \\ -\cos t \end{pmatrix} \right| dt$$

$$= \int_0^{4\pi} \sqrt{(\sin 0.5t)^2 + (\cos t)^2}\, dt$$

(d) $x = -2\cos(0.5t) \quad y = 1 - \sin t$

$\cos t = 2\cos^2(0.5t) - 1$

$= 2\left(\frac{-x}{2}\right)^2 - 1$

$= \frac{x^2}{2} - 1$

$\sin^2 t + \cos^2 t = 1$

$(1 - y)^2 + \left(\frac{x^2}{2} - 1\right)^2 = 1$

$y^2 - 2y + 1 - x^2 + \frac{x^4}{4} = 0$

5. (MSPEC 2020:CA12)

(a) Since at top $t = 65\pi$ sec

$0.2(65\pi) = 40.84$ m

(b) $\underset{\sim}{v}(t) = \begin{pmatrix} -\sin(0.1t) \\ \cos(0.1t) \\ 0.2 \end{pmatrix}$

(c) Speed $= |\underset{\sim}{v}(t)|$

$= \left| \begin{pmatrix} -\sin(0.1t) \\ \cos(0.1t) \\ 0.2 \end{pmatrix} \right|$

$= \sqrt{(-\sin(0.1t))^2 + (\cos(0.1t))^2 + 0.2^2}$

$= \sqrt{1 + 0.2^2}$

$= 1.02$ m/sec

6. (MSPEC 2021:CA19)

(a) $x(t) = \int 32e^{-0.05t} dt = \frac{32e^{-0.05t}}{-0.05} + c$

At $t = 0, x = 100$

Solve $100 = \frac{32e^{o}}{-0.05} + c$

$c = 740$

$\therefore x(t) = 740 - 640e^{-0.05t}$

(b) At $t = 3$, $h = 120 - 2.5(3)^2 = 97.5$ metres so 97.5 metres above horizontal ground.

At $t = 3$, $x = 740 - 640e^{-0.05(3)}$

$= 189.15$ metres.

Sloped ground is given by

$y = 170 - 0.5x$

$y = 170 - 0.5(189.15) = 75.43$ metres.

Height of skier above sloped ground is

$97.5 - 75.43 = 22.07$ metres.

(c) $h = 120 - 2.5t^2$

$\frac{dh}{dt} = -5t$

$\frac{d^2h}{dt^2} = -5 = s - 9.8$

$s = 4.8\ \text{m/s}^2$

(d) From Classpad

Intersection of the curves

$$y = 120 - 1000\left[\ln\left(\frac{740 - x}{640}\right)\right]^2$$

and $y = 120 - 0.5x$ is at (255.92, 42.04) metres.

Using $h = 42.04 = 120 - 2.5t^2$ and solving for t

$t = 5.58$ seconds (or alternatively solve $255.92 = 740 - 640e^{-0.05t}$)

(e) $\underset{\sim}{V}(t) = \begin{pmatrix} 32e^{-0.05t} \\ -5t \end{pmatrix}$

$\underset{\sim}{V}(5.58) = \begin{pmatrix} 24.204 \\ -27.92 \end{pmatrix}$ m/s.

Gradient of sloped ground is −0.5 which can be thought of as the vector $2\underset{\sim}{i} - \underset{\sim}{j}$ from the point of the skier landing down the slope.

Angle between vectors from Classpad, using Action/Vector/Angle ([24.204 − 27.92], [2− 1]) is 22.51°

Chapter 10: Systems of Equations

1. (MSPEC 2016S:CA8)

(a)

$x + y + z = 4 \quad ...R_1$ no change

$2x + 3y + z = 8 \quad ...R_2$ no change

$py + z = p^2 - 1 \quad ...R_3 \quad R_3 \rightarrow 3R_1 - R_3$

$x + y + z = 4 \quad ...R_1$ no change

$y - z = 0 \quad ...R_2 \quad R_2 \rightarrow R_2 - 2R_1$

$py + z = p^2 - 1 \quad ...R_3$ no change

(b)

$x + y + z = 4 \quad ...R_1$

$y - z = 0 \quad ...R_2$

$(p+1)y = p^2 - 1 \quad ...R_3 \quad R_3 = R_2 + R_3$

(c)

$x + y + z = 4$

$y - z = 0$

$(p + 1)\,y = p^2 - 1$

If $p = 1$, $2y = 0$

$y = 0$ giving a unique solution

$x = 4, y = 0, z = 0$

If $p = -1$, $0 \times y = 0$

i.e. $0 = 0$, giving infinitely many solutions

$z = -1$, $\therefore y = -1, x = 6$

2. (MSPEC 2016:CF06)

(a)
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -4 & -2 & -4 \\ 0 & -3 & -1 & -8 \end{array}\right] \begin{array}{l} \\ R_2-3R_1 \\ R_3-2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -4 & -2 & -4 \\ 0 & 0 & 2 & -20 \end{array}\right] \begin{array}{l} \\ \\ 4R_3-3R_2 \end{array}$$

$z = -10$
$y = 6$
$x = 8$

(b) Repeat steps for part (a) except with the k in its position. Row operations as before lead to

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -4 & -2 & -4 \\ 0 & 0 & 4k-2 & -20 \end{array}\right]$$

no solution requires $4k - 2 = 0$

So $k = \frac{1}{2}$

3. (MSPEC 2018:CF2)

(a) Adding last 2 equations
$3x - 2z = 7$ ....④
Adding first 2 equations
$6x - 3z = 9$ ....⑤
Then ⑤ − 2 × ④
$x = -1$
$y = 1$
$z = -5$
Solution is $(-1, 1, -5)$

(b) Unique solution for all values of k except −1. So $k \neq -1$ is unique solution.
When $k = -1$ there will be no solution since the last 2 equations represent parallel planes $x - 2y - z = c$, $c = 2,6$ meaning all 3 planes cannot intersect simultaneously at the same point(s).

4. (MSPEC 2020:CF4)

(a) Not parallel. The co-efficients of the x, y and z terms in any one of the three equations are not multiples of the other two.

(b) $\Pi_1 + \Pi_2$ gives $2x = 11$ $x = 5.5$
$\Pi_1 - \Pi_3$ gives $x = 3$
$\Pi_2 + \Pi_3$ gives $x = 8$
So no solution.

(c) Three non-parallel intersecting planes. Whilst pairs of the planes intersect along a line all 3 planes do not intersect simultaneously at the same points together.

5. (MSPEC 2021:CA14)

(a) $2a + 4c = 108$
$3a + 6c = 162$
$2a + 5c + 2p = 152$

(b) The third equation subtract the first equation is
$1c + 2p = 44$
So the total cost is $44

(c) 3 × Equation 1 gives $6a + 12c = 324$
2 × Equation 2 gives $6a + 12c = 324$
So Equations 1 and 2 represent the same plane. Hence infinite solutions.

(d) Geometric Interpretation Two planes (Equations 1 and 2) which are the same plane being intersected or cut by a third plane (Equation 3) along a line with each point (infinite number) being a solution.
Since only whole dollar amounts for the ticket prices which are positive, there are not really infinite solutions in the context of the problem. Only multiple number of solutions along the line which are whole number co-ordinates.

(e) From part (b): $1c + 2p = 44 \rightarrow p = 22 - 0.5c$
From part (c): $6a + 12c = 324 \rightarrow a = 54 - 2c$

Also given that $a < c < p$ and $a, c, p \; \varepsilon \; z^+$.
Since $a > c$
then $54 - 2c > c$
$54 > 3c$
$18 > c$
Also $c > p$.
So $c > 22 - 0.5c$
$1.5c > 22$
$c > \frac{22}{1.5} = 14.7$
$\therefore c \geq 16$ (must be even otherwise p will not be an integer).
If $c < 18$ and $c \geq 16$ then $c = \$16$ so $a = \$22$ and $p = \$14$.

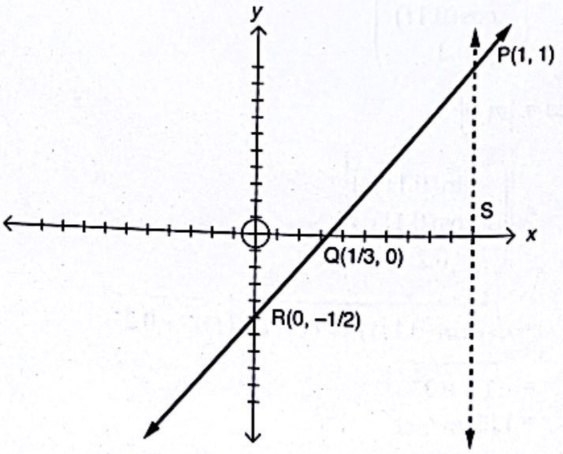
Chapter 11: Implicit Differentiation

1. (3CDMAS 2014:CF5)

$$2y\frac{dy}{dx} = 3x^2 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{2y}$$

At (1,1), $m = \frac{3}{2} \Rightarrow$ Equation of tangent is $y = \frac{3}{2}x - \frac{1}{2}$

Point Q is $(\frac{1}{3}, 0)$ and point R is $(0, -\frac{1}{2})$



$$\Delta PSQ \sim \Delta ROQ \Rightarrow \frac{PQ}{QR} = \frac{SQ}{QO} = 2:1$$

$x^2+y^2=16 \Rightarrow 2x+2y\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{-x}{y}$

$m=y'(2,-2\sqrt{3})=\frac{1}{\sqrt{3}}$

$y=\frac{1}{\sqrt{3}}x+c$ and subst. $(2,-2\sqrt{3})$

$c=-2\sqrt{3}-\frac{2\sqrt{3}}{3}=\frac{-8\sqrt{3}}{3}$

$y=\frac{1}{\sqrt{3}}x-\frac{8\sqrt{3}}{3}$

$\sqrt{3}y=x-8$

3. (MSPEC 2017:CA12a)
Differentiating implicitly with respect to x, we have

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$$

$$m=\frac{dy}{dx}(1,4)=-\frac{\sqrt{4}}{\sqrt{1}}=-2$$

$\therefore y=-2x+c$
$4=-2(1)+c \quad c=6$
$y=-2x+6 \Rightarrow 2x+y=6$
as required

4. (MSPEC 2018:CA20)

(a) $3(x^2+y^2-1)^2\left(2x+2y\frac{dy}{dx}\right)=y^3\cdot 2x+x^2\cdot 3y^2\frac{dy}{dx}$

(b) $3(1+1-1)^2\left(2+2\frac{dy}{dx}\right)=1\cdot 2+1\cdot 3\cdot\frac{dy}{dx}$

$$3\left(2+2\frac{dy}{dx}\right)=2+3\frac{dy}{dx}$$

$$6\frac{dy}{dx}-3\frac{dy}{dx}=2-6$$

$$m=\frac{dy}{dx}=-\frac{4}{3}$$

(c) horizontal gradient at $m=\frac{dy}{dx}=0$

$3(x^2+y^2-1)^2(2x+0)=2xy^3+0$
$6x(x^2+y^2-1)^2=2x^3y$

Solve simultaneously on Classpad with $(x^2+y^2-1)^3=x^2y^3$
(0.514, 1.237)

5. (MSPEC 2020:CF6a,b)

(a) $y=2\tan x$

$\frac{y}{2}=\tan x$

$\tan^{-1}\left(\frac{y}{2}\right)=x$

$\tan^{-1}\left(\frac{x}{2}\right)=y$

$\therefore g(x)=f^{-1}(x)=\tan^{-1}\left(\frac{x}{2}\right)$.

(b) $y=\tan^{-1}\left(\frac{x}{2}\right)$

$2\tan y=x$

$2\sec^2 y\frac{dy}{dx}=1$

$$\begin{aligned}\frac{dy}{dx}&=\frac{1}{2\sec^2 y}\\ \frac{dy}{dx}&=\frac{1}{2(1+\tan^2 y)}\\ &=\frac{1}{2+2\tan^2 y}\\ &=\frac{1}{2+2\tan^2 y}\cdot\frac{2}{2}\\ &=\frac{2}{4+4\tan^2 y}\\ &=\frac{2}{4+(2\tan y)^2}\\ \frac{dy}{dx}&=\frac{2}{4+x^2}\\ &=\frac{a}{x^2+b}\end{aligned}$$

with $a=2, b=4$

6. (MSPEC 2021:CA18)

(a) $3x^2+3y^2\frac{dy}{dx}=3x\cdot 1\frac{dy}{dx}+3y+1\frac{dy}{dx}$

$\frac{dy}{dx}(3y^2-3x-1)=3y-3x^2$

$\frac{dy}{dx}=\frac{3y-3x^2}{3y^2-3x-1}$

(b) $\frac{dy}{dx}=0=\frac{3y-3x^2}{3y^2-3x-1}$

$3y-3x^2=0$
$y=x^2$

Substitute into equation of curve
$x^3+(x^2)^3=3x(x^2)+(x^2)$
$x^3+x^6-3x^3-x^2=0$
$x^2(x^4-2x-1)=0$

$x^2=0$
$x=0$
is the other point apart
from A and B where
the gradient is 0.

or

$x^4-2x-1=0$
Solving on Classpad
for the solution point A,
$x=-0.460$
$y=0.212$

Chapter 12: Related Rates

1. (CA 2008:14)
By Pythagoras' theorem: $l^2=x^2+25^2$
when $l=65$: $65^2=x^2+25^2 \Rightarrow x=60$

Differentiate w.r.t. time $\quad 2l\frac{dl}{dt}=2\times\frac{dx}{dt}$

$$2\times 65\times 1.2=2\times 60\times\frac{dx}{dt}$$

$$\frac{dx}{dt}=1.3$$

The kite is moving at 1.3 m/s

2. (MSPEC 2016S:CA11a)

The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{12}\pi h^3\text{, since } r = \frac{h}{2}$$

$$\frac{dV}{dh} = \frac{1}{4}\pi h^2$$

To find $\frac{dh}{dt}$ when $h = 2.5$

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$$

$$= \frac{4}{\pi h^2} .3$$

$$= \frac{12}{\pi h^2}, h \neq 0$$

$$= \frac{48}{25\pi}\text{, when } h = 2.5$$

The depth is increasing at the rate of $\frac{48}{25\pi}$ m/min when the depth of the ethanol is 2.5 m.

3. (MSPEC 2016:CA15)

(a) V = Area × depth

$$= \frac{2h \cdot h}{2} \times 4$$

$$= 4h^2$$

(b) $\frac{dh}{dt} = \frac{dh}{dv} \cdot \frac{dv}{dt}$

$$= \frac{1}{8h} \cdot (-0.08)$$

$$= \frac{1}{8(0.6)} \cdot (-0.08)$$

$= -0.02$ m/hr (2 d.p.)

(c) $\frac{dh}{dt} = \frac{1}{8h} \cdot \frac{-8}{100}$

$$= -\frac{1}{100h}$$

(d) $\int h\,dh = \int -\frac{1}{100}dt$

$$\frac{h^2}{2} = -\frac{t}{100} + c$$

$t = 0, h = \frac{1}{\sqrt{2}}$ (max height)

$$c = \frac{1}{4}$$

$$h = \sqrt{\frac{1}{2} - \frac{t}{50}}$$

4. (MSPEC 2017:CA18)

(a) 1 revolution every 12 seconds

$\therefore \frac{2\pi}{12} = \frac{\pi}{6}$ radians/sec

(b) Using cosine rule

$s^2 = 5^2 + 8^2 - 2(5)(8)\cos\theta$

$s^2 = 89 - 80\cos\theta$

(c) Differentiate implicitly with respect to time

$$2s\frac{ds}{dt} = 80\sin\theta \bullet \frac{d\theta}{dt}$$

$$\frac{ds}{dt} = \frac{80\sin\theta \bullet \frac{\pi}{6}}{2s}$$

at $t = 4$ $\theta = 4 \times \frac{\pi}{6} = \frac{2\pi}{3}$

$$\frac{ds}{dt} = \frac{\frac{80\pi}{6}\sin\left(\frac{2\pi}{3}\right)}{2\sqrt{89 - 80\cos\left(\frac{2\pi}{3}\right)}}$$

$= 1.60$ m/sec

(d) Sketch $y = \frac{ds}{dt} = \frac{\frac{80\pi}{6}\sin\theta}{2\sqrt{89 - 80\cos\theta}}$ on Classpad and find co-ordinates of maximum value.

First max. @ (0.8957, 2.618)

$\theta = 0.8957$

$\therefore \cos\theta = \cos 0.8957$

$= 0.625$ (or $\frac{5}{8}$)

5. (MSPEC 2020:CA20b)

Using x as defined

$$\frac{dx}{dt} = 0.2$$

Area of $PQRS$ is given by considering the length of the side of this square.

By Pythagoras

$$PS = \sqrt{x^2 + (10 - x)^2}$$

$$\text{Area} = \left(\sqrt{x^2 + (10 - x)^2}\right)^2$$

$$A = x^2 + (10 - x)^2$$

$$\frac{dA}{dx} = 2x + 2(10 - x)(-1) = 4x - 20$$

$$\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt}$$

at $x = 3$

$\frac{dA}{dt} = (-8)(0.2) = -1.6$ cm²/sec

6. (MSPEC 2021:CA9)

Speed of beam of light is $\frac{dx}{dt}$.

Now $\frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt}$

Given → $\frac{d\theta}{dt} = 6\pi$ rads/min (3 revolutions/min)

$= \frac{6\pi}{60} = \frac{\pi}{10}$ rads/sec

Also $\tan\theta = \frac{x}{50}$

$$x = 50\tan\theta$$

$$\frac{dx}{d\theta} = 50\sec^2\theta$$

$$\therefore \frac{dx}{dt} = \frac{50}{\cos^2\theta} \cdot \frac{\pi}{10}$$

at 100 metres up the coast $x = 100 \Rightarrow \tan\theta = \frac{100}{50}$

$$\theta = \tan^{-1}(2)$$

$$\frac{dx}{dt} = \frac{50}{(\cos(\tan^{-1}2))^2} \cdot \frac{\pi}{10}$$

$$= 78.54 \text{ m/sec}$$

Chapter 13: Integration Techniques

1. (3CDMAS 2011:CF6)

Let $u = 2t^3 + t + 1 \Rightarrow dt = \dfrac{du}{6t^2 + 1}$

When $t = 0$, $u = 1$ and when $t = 10$, $u = 2011$.
The required integral, I, is thus

$$I = \int_1^{2011} \frac{1}{u^{\frac{1}{2}}}\,du = [2\sqrt{\text{u}}]_1^{2011} = 2\sqrt{2011} - 2$$

2. (MSPEC 2016:CF04)

(a) $\dfrac{a(x-3) + b(x+2)}{(x+2)(x-3)}$

$= \dfrac{(a+b)x + 2b - 3a}{(x+2)(x-3)}$

$\Rightarrow a + b = 1 \qquad 2b - 3a = -8$

$\qquad a = 2 \qquad\qquad b = -1$

$\therefore \dfrac{2}{x+2} - \dfrac{1}{x-3}$

(b) $\displaystyle\int\left(\frac{2}{x+2} - \frac{1}{x-3}\right) dx$

$= 2 \ln |x+2| - \ln |x-3| + c$

$= \ln \left|\dfrac{(x+2)^2}{x-3}\right| + c$

3. (MSPEC 2016:CA09)

$u = 1 + x \Rightarrow u - 1 = x$

$du = dx$

$$I = \int (u-1)u^{\frac{n}{2}}\,du$$

$$= \int\left(u^{\frac{n+2}{2}} - u^{\frac{n}{2}}\right) du$$

$$= \frac{u^{\frac{n+4}{2}}}{\frac{n+4}{2}} - \frac{u^{\frac{n+2}{2}}}{\frac{n+2}{2}} + c$$

$$= \frac{2\sqrt{(1+x)^{n+4}}}{n+4} - \frac{2\sqrt{(1+x)^{n+2}}}{n+2} + c$$

4. (MSPEC 2017:CA16b)

$$I = \int_{\frac{a}{2}}^{a} f(2x-a)\,dx$$

$$= \int_0^a f(u)\frac{du}{2}$$

$$= \frac{1}{2}\int_0^a f(u)\,du$$

Since $\displaystyle\int_0^a f(u)\,du$ is the area of the triangle with base 'a' and height 'b'

$$I = \frac{1}{2} \bullet \frac{a \times b}{2}$$

$$= \frac{ab}{4}$$

5. (MSPEC 2018:CF6)

(a) $2 = a(x+1) + b(x-1)$.

$x = 1 \Rightarrow a = 1$

$x = -1 \Rightarrow b = -1$

(b) $= \dfrac{1}{2}\displaystyle\int \frac{2}{x^2 - 1}\,dx$

$= \dfrac{1}{2}\displaystyle\int \frac{1}{x-1} - \frac{1}{x+1}\,dx$

$= \dfrac{1}{2}\,[\ln|x-1| - \ln|x+1|] + c$

$= \dfrac{1}{2}\ln\left|\dfrac{x-1}{x+1}\right| + c$

6. (MSPEC 2019:CF3)

(a) $2x^2 + 5x + 6 = ax(x+3) + b(x+3) + cx^2$

$2x^2 + 5x + 6 = (a+c)x^2 + (3a+b)x + 3b$

$a + c = 2 \qquad 3a + b = 5 \qquad 3b = 6$

Solving

$b = 2 \quad a = 1 \quad c = 1$

(b) $= \displaystyle\int \frac{1}{x} + \frac{2}{x^2} + \frac{1}{x+3}\,dx$

$= \ln|x| - \dfrac{2}{x} + \ln|x+3| + c$

7. (MSPEC 2020:CF6c,d)

(a) $\dfrac{q}{x^2+4} + \dfrac{r}{x-3} = \dfrac{q(x-3) + r(x^2+4)}{(x^2+4)(x-3)}$

$= \dfrac{qx - 3q + rx^2 + 4r}{(x^2+4)(x-3)}$

$= \dfrac{rx^2 + qx + 4r - 3q}{(x^2+4)(x-3)}$

$= \dfrac{3x^2 + 2x + 6}{(x^2+4)(x-3)}$

with $r = 3 \quad q = 2, \quad 4r - 3q = 6$

(b) $= \displaystyle\int \frac{2}{x^2+4} + \frac{3}{x-3}\,dx$

$= \tan^{-1}\left(\dfrac{x}{2}\right) + 3\ln|x-3| + c$

8. (MSPEC 2020:CF7)

$u = (x+2)^{\frac{1}{2}} \qquad u^2 - 2 = x$

$du = \dfrac{1}{2}(x+2)^{-\frac{1}{2}}\,dx$

$du = \dfrac{1}{2\sqrt{x+2}}\,dx$

$du = \dfrac{1}{2u}\,dx$

$2u\,du = dx$

$= 3\int_1^3 \frac{u^2-2}{u} \cdot 2u\,du$

$= 6\int_1^3 u^2 - 2\,du$

$= 6\left[\frac{u^3}{3} - 2u\right]_1^3$

$= 6\left[\left(\frac{27}{3} - 6\right) - \left(\frac{1}{3} - 2\right)\right]$

$= 28$

9. (MSPEC 2021:CF3)

Let $u = x - 2$ $\quad x = 2 \rightarrow u = 0$

$\frac{du}{dx} = 1$ $\quad x = 3 \rightarrow u = 1$

$= 15\int_0^1 (u+2)u^{\frac{1}{2}} \frac{dx}{du} \cdot du$

$= 15\int_0^1 \left(u^{\frac{3}{2}} + 2u^{\frac{1}{2}}\right) du$

$= 15\left[\frac{2u^{\frac{5}{2}}}{5} + \frac{4u^{\frac{3}{2}}}{3}\right]_0^1$

$= 26$

10. (MSPEC 2021:CF5)

(a) $7x^2 - 12x + 2 = a(x^2 + 2) + bx(x - 2)$

$= (a + b)x^2 - 2bx + 2a$

$2a = 2$

$a = 1$

$2b = 12$

$b = 6$

(b) $= \int \frac{1}{x-2} + \frac{6x}{x^2+2}\,dx$

$= \ln|x - 2| + 3\ln(x^2 + 2) + c$

$= \ln\left|(x-2)(x^2+2)^3\right| + c$

Chapter 14: Integration Techniques and Trigonometric Functions

1. (MSPEC 2016:CF05)

(a) $du = 2\cos 2x\,dx$

$\int_0^1 6u^4\,du$

$= \left[\frac{6u^5}{5}\right]_0^1$

$= \frac{6}{5}$

(b) $\int_0^{\frac{1}{2}} \sec^2\left(\frac{\pi x}{2}\right) - 1\,dx$

$= \left[\frac{2}{\pi}\tan\left(\frac{\pi x}{2}\right) - x\right]_0^{\frac{1}{2}}$

$= \frac{2}{\pi} - \frac{1}{2}$

2. (MSPEC 2017:CF3)

(a) $I = \int_0^{\frac{\pi}{4}} \frac{\tan^2 u}{(1+\tan^2 u)^2} \sec^2 u\,du$

$= \int_0^{\frac{\pi}{4}} \frac{\tan^2 u}{\sec^2 u}\,du$

$= \int_0^{\frac{\pi}{4}} \frac{\sin^2 u}{\cos^2 u} \bullet \cos^2 u\,du$

$= \int_0^{\frac{\pi}{4}} \sin^2 u\,du$ (as required)

$a = 0$ $\; b = \frac{\pi}{4}$

(b) $\frac{1}{2}\int_0^{\frac{\pi}{4}} 1 - \cos 2u\,du$

$= \frac{1}{2}\left[u - \frac{\sin 2u}{u}\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{8} - \frac{1}{4}$

3. (MSPEC 2018:CF5)

$u = \cos(2x)$

$du = -2\sin(2x)dx$

$x = 0 \quad u = 1$

$x = \frac{\pi}{4} \quad u = 0$

$\therefore -\frac{1}{2}\int_1^0 u^{1008}\,du$

$= \frac{1}{2}\int_0^1 u^{1008}\,du$

$= \frac{1}{2}\left[\frac{u^{1009}}{1009}\right]_0^1$

$= \frac{1}{2018}$

4. (MSPEC 2018:CF9)

(a) $(\sqrt{3}\tan(u) + 1)^2 - 2(\sqrt{3}\tan(u) + 1) + 4$

$3\tan^2(u) + 2\sqrt{3}\tan(u) + 1 - 2\sqrt{3}\tan(u) - 2 + 4$

$3\tan^2(u) + 3$

$3(1 + \tan^2(u))$

$3\sec^2(u)$

(b) Let $x = \sqrt{3}\tan(u)$

$$dx = \sqrt{3}\sec^2(u)\,du$$

$x = 1 \quad u = 0$

$x = 2 \quad u = \frac{\pi}{6}$

$$= \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}\sec^2(u)du}{(3\sec^2(u))^{\frac{3}{2}}}$$

$$= \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}\sec^2 u du}{3\sqrt{3}\sec^2(u)\sqrt{\sec^2(u)}}$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{6}} \cos u\, du$$

$$= \frac{1}{3}[\sin u]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{6}$$

5. (MSPEC 2019:CF1)

$$3\int_0^{\frac{\pi}{2}} \sin\left(\frac{5x}{2} + \frac{x}{2}\right) + \sin\left(\frac{5x}{2} - \frac{x}{2}\right)\, dx$$

$$= 3\int_0^{\frac{\pi}{2}} \sin 3x + \sin 2x\, dx$$

$$= 3\left[\frac{-\cos 3x}{3} - \frac{\cos 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 3\left[\left(0 - \frac{1}{2}\right) - \left(\frac{1}{3} + \frac{1}{2}\right)\right]$$

$$= 4$$

6. (MSPEC 2019:CF6)

$dx = 2\cos\theta\, d\theta$

$x = 0 \quad \theta = 0$

$x = \sqrt{3} \quad \theta = \frac{\pi}{3}$

$$= \int_0^{\frac{\pi}{3}} \sqrt{1 - \frac{(2\sin\theta)^2}{4}}\, .\, 2\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{1 - \sin^2\theta}\, .\, 2\cos\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{3}} \cos^2\theta\, d\theta$$

$$= \frac{2}{2}\int_0^{\frac{\pi}{3}} 1 + \cos 2\theta\, d\theta$$

$$= \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{4}$$

7. (MSPEC 2020:CF1)

Since $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$$= \frac{4}{2}\int_0^{\pi}(1 + \cos 2x)dx - \int_0^{\pi}\sin x\, dx$$

$$= 2\left[x + \frac{\sin 2x}{2}\right]_0^{\pi} + [\cos x]_0^{\pi}$$

$$= 2\left[\left(\pi + \frac{\sin 2\pi}{2}\right) - (0 + 0)\right] + [\cos x - \cos 0]$$

$$= [2\pi] + [-2]$$

$$= 2\pi - 2$$

Chapter 15: Area and Volume

1. (MSPEC 2016:CA13)

(a) $2 = \cos y\, \frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{2}{\cos y} \text{ or } 2\sec y$$

(b) $A = \int_0^{\pi} \frac{\sin y}{2}\, dy$

$$= \left[-\frac{1}{2}\cos y\right]_0^{\pi}$$

$$= -\frac{1}{2}[\cos\pi - \cos 0]$$

$$= 1 \text{ unit}^2$$

2. (MSPEC 2016:CA17b,c)

(a) X intercept at C ($y = 0$)

$0 = x - 8$

$x = 8 \Rightarrow (8, 0)$

(b) $V = \pi\int_{-4}^{2} 16 - x^2\, dx + \pi\int_2^8 \left(\frac{x-8}{\sqrt{3}}\right)^2 dx$

$$= 96\pi$$

$$= 301.59 \text{ units}^3 \text{ (2 d.p.)}$$

3. (MSPEC 2017:CF8)

(a) Use $V_y = \pi\int_a^b x^2\, dy$

Now $y = \left(\frac{x-a}{b-a}\right) h$

$$\frac{b-a}{h}\, y = x - a$$

$$\frac{b-a}{h}\, y + a = x$$

$$\therefore V = \pi\int_0^h \left[\frac{b-a}{h}\, y + a\right]^2 dy$$

(b) $V = \left.\dfrac{\pi\left[\frac{b-a}{h}\, y + a\right]^3}{3\frac{b-a}{h}}\right|_0^h$

$$= \frac{\pi h}{3(b-a)}\left(\left[\frac{b-a}{h} \bullet h + a\right]^3 - [a]^3\right)$$

$$= \frac{\pi h}{3(b-a)}(b^3 - a^3)$$

or continuing simplifying

$$= \frac{\pi h}{3(b-a)}(b-a)(b^2 + ab + a^2)$$

$$= \frac{\pi h}{3}(a^2 + ab + b^2)$$

4. (MSPEC 2017:CA12b)

Shaded area is the area between the curve and the x-axis from $x = 1$ to $x = 9$ (x intercept) subtract the area between the tangent and the x-axis from $x = 1$ to $x = 3$ (x intercept)

$$A = \int_1^9 (3 - \sqrt{x})^2\,dx - \int_1^3 6 - 2x\,dx$$

= 4 units^2 (using Classpad)

5. (MSPEC 2018:CA15)

(a) $3x^2 + 8\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-3x^2}{8}$$

$$m = \frac{-3(2)^2}{8} = -1.5$$

$y = -1.5x + c$

$7 = -1.5(2) + c$

$c = 10$

$\therefore y = -1.5x + 10$

(b) On Classpad solve

$$\frac{64 - x^3}{8} = -1.5x + 10$$

$x = -4, 2$

$$\text{Area} = \int_{-4}^{2} (-1.5x + 10) - \left(\frac{64 - x^3}{8}\right) dx$$

$= 13.5$

$$\text{(c) Volume} = \pi \int_{-4}^{2} (-1.5x + 10)^2 - \left(\frac{64 - x^3}{8}\right)^2 dx$$

$= 920.94$

6. (MSPEC 2019:CF8)

(a) $V = \pi \int_0^5 x^2\,dy$

Now $x^2 = y^2(36 - x^2y)$

$x^2 + x^2y^3 = 36y^2$

$x^2(1 + y^3) = 36y^2$

$$x^2 = \frac{36y^2}{1 + y^3}$$

$$\therefore V = \pi \int_0^5 \frac{36y^2}{1 + y^3}\,dy$$

(b) $V = 12\pi \int_0^5 \frac{3y^2}{1 + y^3}\,dy$

$= 12\pi\ [\ln|1 + y^3|]_0^5$

$= 12\pi\ [\ln 126 - \ln 1]$

$= 12\pi \ln 126$ cm^3

7. (MSPEC 2020:CA9)

$$Vy = \pi \int_a^b [f(y)]^2\,dy \quad x = \sin\left(\frac{y}{n}\right) + 3.$$

$$V = \pi \int_0^6 \left(\sin\left(\frac{y}{n}\right) + 3\right)^2 dy = 259.53 \text{ cm}^3$$

80% of the volume = 0.8 × 259.53 = 66.089

Solve for h $\quad 66.089 = \pi \int_0^h \left(\sin\left(\frac{y}{n}\right) + 3\right)^2 dy$ on Classpad.

$$h = 4.96 \text{ cm}$$

8. (MSPEC 2020:CA16)

(a) $y - 1 = \pm\sqrt{\sin\left(\frac{\pi x}{2}\right)}$

Upper semi circle is $y = 1 + \sqrt{\sin\left(\frac{\pi x}{2}\right)}$

Area $= 2\int_0^2 \left(1 + \sqrt{\sin\left(\frac{\pi x}{2}\right)} - 1\right) dx$ (<u>Note</u> $y = 1$ is the curve below)

$= 3.0510$ units^2

(b) $A_0 = \pi r^2$

$A = \pi \times 1^2$

$A = \pi = 3.14159$ units^2

Areas in (a) and (b) are close but not equal. So shape is not a circle.

9. (MSPEC 2021:CF8)

(a) $(y - \sqrt{x})^2 = 2 - x^2$

$$y - \sqrt{x} = \pm\sqrt{2 - x^2}$$

$$y = \sqrt{x} \pm \sqrt{2 - x^2}$$

$\therefore x \geq 0$ and $2 - x^2 \geq 0$

$$2 \geq x^2$$

$$-\sqrt{2} \leq x \leq \sqrt{2}$$

So domain is $0 \leq x \leq \sqrt{2}$

(b) The upper curve of the right-hand side of the heart is given by $y_u = \sqrt{x} + \sqrt{2 - x^2}$.

The lower curve is given by $y_L = \sqrt{x} - \sqrt{2 - x^2}$.

So using $A = 2\int_0^{\sqrt{2}} (y_u - y_L)\,dx$

$$= 2\int_0^{\sqrt{2}} \sqrt{x} + \sqrt{2 - x^2} - \left(\sqrt{x} + \sqrt{2 - x^2}\right) dx$$

$$= 2\int_0^{\sqrt{2}} 2\sqrt{2 - x^2}\,dx = 4\int_0^{\sqrt{2}} \sqrt{2 - x^2}\,dx$$

<u>Note:</u> Could also use the sum of the areas between the curves of the heart and $y = \sqrt{x}$

i.e. $2\int_0^{\sqrt{2}} ((y_u - \sqrt{x}) + (\sqrt{x} - y_L))\,dx$

(c) $\text{Area} = 4\int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx$

$x = \sqrt{2}\sin\theta \qquad x = 0 \Rightarrow \theta = 0$

$\frac{dx}{d\theta} = \sqrt{2}\cos\theta \qquad x = \sqrt{2} \Rightarrow \theta = \frac{\pi}{2}$

$$\text{Area} = 4\int_0^{\frac{\pi}{2}} \sqrt{2-(\sqrt{2}\sin\theta)^2} \cdot \frac{dx}{d\theta} \cdot d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{2(1-\sin^2\theta)} \cdot \sqrt{2}\cos\theta\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} (\sqrt{2})^2 \cos\theta \cdot \cos\theta\,d\theta$$

$$= 8\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= \frac{1}{2} \cdot 8\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta$$

$$= 4\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 4\left[\left(\frac{\pi}{2} + \frac{\sin\pi}{2}\right) - (0)\right]$$

$= 2\pi$ units^2

10. (MSPEC 2021:CA13)

(a) Centre is (10, 0) radius of 10

so $(x-10)^2 + y^2 = 10^2$

$x^2 - 20x + 100 + y^2 = 100$

$x^2 + y^2 = 20x$ as required.

(b) $V_x = \pi\int_0^h y^2\,dx$

$$= \pi\int_0^h (20x - x^2)\,dx$$

$$= \pi\left[10x^2 - \frac{x^3}{3}\right]_0^h$$

$$= \pi\left(10h^2 - \frac{h^3}{3}\right)$$

Chapter 16: Differential Equations

1. (MSPEC 2016:CA18)

(a) max T.P. at $x = 2$ and slope field has symmetry and shape of parabola so $y = a(x-2)^2 + b$, $a < 0$

But general differential equation required so

$\frac{dy}{dx} = 2a(x-2)$, $a < 0$

(b) $\frac{dy}{dx} = 2a(x-2)$

$0.5 = 2a(1-2)$

$a = -\frac{1}{4}$

$y = -\frac{1}{4}(x-2)^2 + b$

$2 = -\frac{1}{4}(1-2)^2 + b$

$b = \frac{9}{4}$

So $y = -\frac{1}{4}(x-2)^2 + \frac{9}{4}$

2. (MSPEC 2017:CA11)

(a) $\frac{dy}{dx}(0,1) = \frac{1}{2(0)-2} = -\frac{1}{2}$

(b) Vertical, straight up and down

(c) $\frac{dy}{dx} = \frac{1}{2(x-1)}$

$\int 2dy = \int \frac{dx}{x-1}$

$2y = \ln|x-1| + c$

put $x = 0$, $y = 1$ into function

$c = 2$

$\therefore y = \frac{1}{2}\ln|x-1| + 1$

3. (MSPEC 2017:CA17c,d)

(a) $\frac{dA}{dt} = -0.4A$

So $A = A_0e^{-0.4t}$ is solution

$A_0 = 8$

$\therefore A = 8e^{-0.4t}$

(b) Solve

$0.01 = 8e^{-0.4t}$

$t = 16.71$ seconds

Appears to stop after 16.8 seconds

4. (MSPEC 2018:CA18)

(a) $60 = k(100)(1600-100)$

$\frac{60}{150\,000} = k$

$k = 0.0004$ as required

(b) $\delta N \approx \frac{dN}{dt} \cdot \delta t$

$= 0.0004(500)(1600-500) \cdot 0.25$

$= 55$ people

(c) $\frac{dN}{dt} = 0.0004N(1600-N)$

$$\int \frac{dN}{N(1600-N)} = \int 0.0004\,dt$$

$$\frac{1}{1600}\int \frac{1}{N} + \frac{1}{1600-N}\,dN = \int 0.0004\,dt$$

$$\frac{1}{1600}\left[\ln|N| - \ln|1600-N|\right] = 0.0004t + c$$

since $N \geq 0$ and $1600 - N \geq 0$ (max N is 1600)

$$\ln\left(\frac{N}{1600-N}\right) = 0.64t + b$$

$$\frac{N}{1600-N} = Ae^{0.64t} \quad (A = e^b)$$

$t = 0 \quad N = 100$

$$\frac{100}{1500} = A = \frac{1}{15}$$

$$\therefore \frac{N}{1600-N} = \frac{e^{0.64t}}{15}$$ as required

(d) Fastest rate at $\frac{d^2N}{dt^2} = 0$

which occurs at $N = 800$ (half way to 1600)
Solve on Classpad

$$\frac{800}{1600-800} = \frac{e^{0.64t}}{15}$$

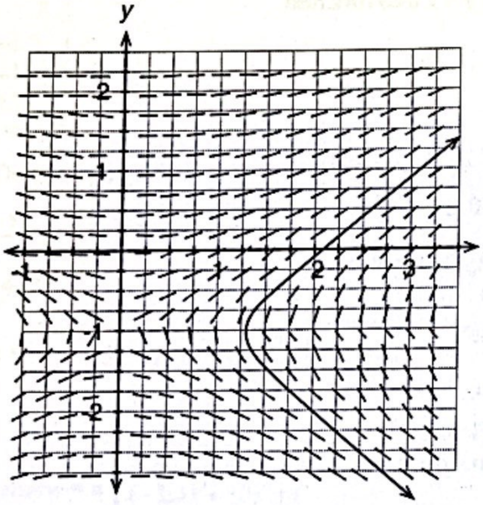
or $\frac{\ln 15}{64} = t = 4.2313$ hours

At 12.14 pm

5. (MSPEC 2019:CA11)

(a) $\frac{dy}{dx} = \frac{2}{0(2)+1} = 1$

(b)



(c) $\frac{dy}{dx} = \frac{x}{2y+2}$

$\int 2y + 2\,dy = \int x\,dx$

$$y^2 + 2y = \frac{x^2}{2} + c$$

Subst. (2, 0) gives $c = -2$

$$\therefore \ y^2 + 2y = \frac{x^2}{2} - 2$$

or $2y^2 + 4y - x^2 + 4 = 0$

6. (MSPEC 2019:CA17)

(a) $\frac{dP}{dt} = 0.1\,P$

so $P = P_0e^{0.1t}$

$30\,000 = P_0e^{0.1(55)}$

$P_0 = 123$ whales

(b) a is the limiting size $= \frac{0.1}{\frac{1}{700\,000}} = 70\,000$

$c = 0.1$

$$30\,000 = \frac{70\,000}{1 + be^0}$$

$$b = \frac{4}{3}$$

(c) Solve $60\,000 = \frac{70\,000}{1 + \frac{4}{3}e^{-0.1t}}$

$t = 20.79$
Year is –End of year 2018 + 20.79
so during the year 2039
(d) In part (a) $P(t)$ continues to increase exponentially without bound
In (b) $P(t)$ continues to increase but there is a limiting size (bound) of 70 000 whales that can never be exceeded

7. (MSPEC 2020:CA19)
(a) $t = 0 \quad P = 0.01$
Size of population is 0.01 million, so 10 000 tonnes.
(b) k is the limiting size, $k = 2.4$

$r = \frac{0.3}{2.4} = \frac{1}{8}$ or 0.125

$$\frac{dp}{dt} = \frac{1}{8}P(2.4 - P)$$

(c) $P = 0.5$

Incremental Formula $\delta p \approx \frac{dp}{dt} \cdot \delta t$

Using part (b) for $\frac{dp}{dt}$

$$\delta p \approx \frac{1}{8}(0.5)(2.4 - 0.5) \cdot \frac{1}{12}$$
$$= 0.009896$$

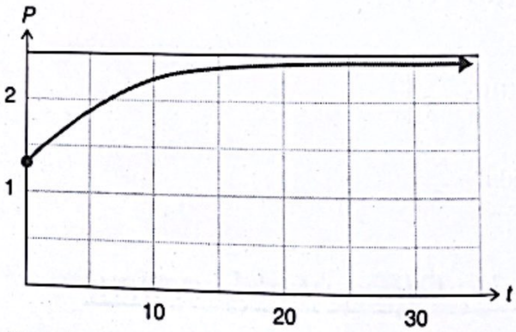
So 9896 tonnes.

(d) Maximum rate of growth at halfway point between 0 and limiting size.
So at $P = 1.2$

$$\frac{dp}{dt} = \frac{1}{8} \cdot (1.2)(2.4 - 1.2)$$
$$= 0.18$$

So 180 000 tonnes/year.

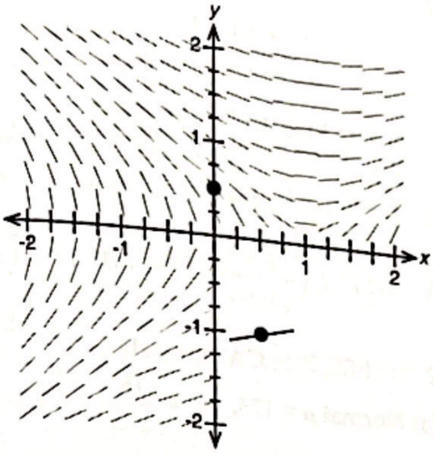
(e)



Starting from $P = 1.3$ and plot points from original graph shifted across eg. $t = 30$ on original graph is approx. 12 years after the point of inflection I which is close to our initial population of $P = 1.3$
$\therefore P$ at $t = 30$ (on original) is $t \approx 12$ (on this graph).

(a) $\frac{dy}{dx} = \frac{0.5-1}{2(-1)}$

$= \frac{1}{4}$



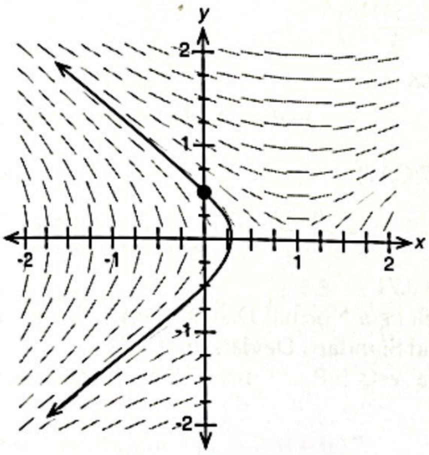
(b) $\int 2y\,dy = \int (x-1)\,dx$

$y^2 = \frac{x^2}{2} - x + c$

$(0.5)^2 = c = \frac{1}{4}$

$\therefore y^2 = \frac{x^2}{2} - x + \frac{1}{4}$

(c)



Chapter 17: Modelling Motion

1. (MSPEC 2016:CA11)

(a) $v = \frac{kt^2}{2} + c$

$t = 0, v = 0 \Rightarrow c = 0$

$v = \frac{kt^2}{2}$

$t = 4, v = 1.2$

$1.2 = \frac{k \cdot 4^2}{2}$

$\frac{2.4}{16} = k$

$k = \frac{24}{160} = \frac{3}{20}$

(b) $\delta v \approx \frac{dv}{dt} \cdot \delta t$

$= \frac{3}{20} t \cdot \delta t$

$= \frac{3}{20}(2)(0.1)$

$= \frac{3}{100}$ or 0.03 m/s

(c) Distance $= 2\int_0^4 \frac{3t^2}{40}\,dt + (12 \times 1.2)$

$= 17.6$ m

2. (MSPEC 2017:CA17a,b)

(a) SHM $\ddot{x} = -k^2 x$

with $k = \pi$

Since starts at origin and moves right

$x = A\sin(\pi t) \quad (A > 0)$

$v = \pi A\cos(\pi t)$

$t = 0 \quad v = 8\pi$

$8\pi = \pi A$

$A = 8$

$\therefore x = 8\sin(\pi t)$

(b) Distance $= \int_0^5 |8\pi\cos(\pi t)|\,dt$

$= 80$ cm

3. (MSPEC 2018:CA13)

(a) $a = v \cdot \frac{dv}{dx}$

$= v\,(-2x^{-2})$

$= \frac{2}{x}\left(\frac{-2}{x^2}\right)$

$= \frac{-4}{x^3}$

(b) $\frac{dx}{dt} = \frac{2}{x}$

$\int x\,dx = \int 2\,dt$

$\frac{x^2}{2} = 2t + c$

$t = 0 \quad x = 2$

$\frac{2^2}{2} = 0 + c \Rightarrow c = 2$

$\therefore x^2 = 4t + 4$

$x = \pm\sqrt{4t+4}$

$x = \sqrt{4t+4}$ (reject –ve since $t = 0$, $x = +2$)

$x = 2\sqrt{t+1}$

4. (MSPEC 2019:CA18)

(a) Goes through 2π radians in 72 seconds

$\frac{2\pi}{72} = \frac{\pi}{36}$ rads/sec

(b) At the start of the ride no rotation yet so $\theta = 0$ rads and $y(0) = -80$

Substituting

$-80 = 80\sin(0 + a)$

$-1 = \sin a$

$a = \frac{-\pi}{2}$

(c) 100 m above means $y = 20$

Solve $80\sin\left(\theta - \frac{\pi}{2}\right) = 20$

$\theta = 1.8235$ rads

Now $\frac{dy}{dt} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dt}$

$= 80\cos\left(\theta - \frac{\pi}{2}\right) \cdot \frac{\pi}{36}$

at $\theta = 1.8235\ \frac{dy}{dt} = 6.76$ m/s

(d) $\theta = \frac{\pi}{36}t$ so $y(t) = 80\sin\left(\frac{\pi t}{36} - \frac{\pi}{2}\right)$

$$y'(t) = \frac{\pi}{36} \cdot 80\cos\left(\frac{\pi t}{36} - \frac{\pi}{2}\right)$$

$$y''(t) = \frac{-\pi^2}{36^2} \cdot 80\sin\left(\frac{\pi t}{36} - \frac{\pi}{2}\right)$$

$$y'' = \frac{-\pi^2}{36^2} y \text{ which is SHM}$$

(e) Will be the situation at $t = \frac{7}{16}$ of 72 seconds = 31.5 sec
Require $y'(31.5)$

$$y'(t) = \frac{80\pi}{36}\cos\left(\frac{\pi t}{36} - \frac{\pi}{2}\right)$$

at $t = 31.5 \quad y' = 2.67$ m/s

5. (MSPEC 2020:CA14)

(a) $a = v\frac{dv}{dx}$

Now $\frac{dv}{dx} = -0.2$

$\therefore a = v(-0.2) = -0.2v = -0.2(-0.2\,x)$
$= 0.04x$

(b) No, for SHM $a = -k^2x$. This is not possible since $a = 0.04x$ cannot be expressed in the form $a = -k^2\,x$.

(c) $t = 0 \Rightarrow x = 4$
$v = -0.2x$

$\frac{dx}{dt} = -0.2x$

$\int \frac{dx}{x} = \int -0.2\,dt$

$\ln|x| = -0.2t + c$
$\ln 4 = c$
$\therefore \ln|x| = -0.2t + \ln 4$
$\ln|x| - \ln 4 = -0.2t$
$\ln\left|\frac{x}{4}\right| = -0.2t$
$x = 2$
$\ln\left|\frac{2}{4}\right| = -0.2t$
$t = 3.47$ sec

<u>Note</u> $\frac{x}{4} = e^{-0.2t} \quad (e^{-0.2t} > 0)$
$x = 4e^{-0.2t}$

6. (MSPEC 2021:CA12)

(a) $n = \frac{2\pi}{60} = \frac{\pi}{30}$

$x = A\cos\left(\frac{\pi}{30}t\right)$

$v = -\frac{\pi A}{30}\sin\left(\frac{\pi}{30}t\right)$

$\therefore \frac{\pi A}{30} = \frac{\pi}{2}$

$A = 15$ metres

(b) $a = -\left(\frac{\pi}{30}\right)^2 X$ (SHM equation)

$a = -\left(\frac{\pi}{30}\right)^2 \cdot 10$

$= -0.110$ m/s^2

Chapter 18: Sample Means and Probability Distributions

1. (3CDMAT 2010:CA18a,d)
(a) $T \sim N(12.2, 2.5^2) \Rightarrow P(T > 16) = 0.0643$

(b) Let $\overline{X}$ denote the average burn time from a random sample of 1000 matches.
$\overline{X} \sim N\left(12.2, \left(\frac{2.5}{\sqrt{1000}}\right)^2\right) \Rightarrow P(12.15 \le \overline{X} \le 12.25) = 0.4729$

2. (MSPEC 2016:CA19a,b,c,d)

(a) Normal $\mu = 175$, $\sigma_{\overline{X}} = \frac{15}{\sqrt{50}}$

$P(173 < \overline{X} < 177) = 0.6542$

(b) total 8.96 kL $\Rightarrow \frac{8960}{50} = 179.2$ per top up

$P(\overline{X} < 179.2) = 0.9761$

(c) Inverse Normal same μ and σ as parts (a) and (b) from Classpad $a = 169.5$, $b = 180.5$ (1 d.p.)

(d) Use $n > \left(\frac{Z \cdot \sigma}{\text{Tolerance}}\right)^2$

$n > \left(\frac{(2.054)15}{5}\right)^2$

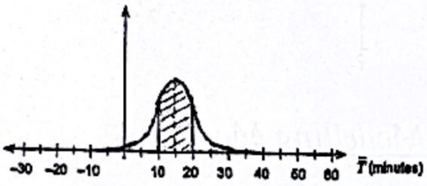
$n > 37.96$
$n = 38$

3. (MSPEC 2017:CA9)

(a) $\overline{T} \sim N\left(\mu = 15, \sigma_{\overline{T}} = \frac{\sqrt{675}}{\sqrt{30}}\right)$

$P(10 \le \overline{T} \le 20) = 0.71$

(b) Diagram will be a Normal Distribution bell curve with a mean of 15 and Standard Deviation of 4.74.



4. (MSPEC 2018:CA12)

(a) The sample means ($n \ge 30$) are always Normally distributed $\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$ no matter what the parent distribution is (in this case exponential distribution) provided sample is randomly chosen.

$\therefore \overline{X} \sim N\left(20, \left(\frac{20}{\sqrt{50}}\right)^2\right)$

(b) $P(15 < \overline{X} < 25) = 0.9229$

(c) Will not change since it doesn't matter what the parent distribution (whether exponential or a more complicated one) the sample means are always assumed to be Normally distributed and the parameters only dependent on μ and σ which are unchanged along with $n = 50$. The sample size of 50 ($n > 30$) confirms the Normal Distribution.

(d) $P(\overline{X} > 25) = 0.03$ (Given)
We require the sampling standard deviation that was used in the Normal Distibution to get the probability above.

On Classpad (or E activity)
Solve (norm CDf $(25, \infty, x, 20) = 0.03$
$x = 2.65845$ (Sampling Std Deviation)

Since $x = \dfrac{\sigma}{\sqrt{n}}$ and $\sigma = 20$

$$n = 57\ (56.6)$$

5. (MSPEC 2019:CA14)

(a) $\bar{T}$ is approx. Normal
mean is 80

Std. dev. $= \dfrac{20}{\sqrt{100}} = 2$

(b) $P(\bar{T} > 83) = 0.0668$
(c) Smaller answer.
The greater the value of n, the smaller the standard deviation. The Normal bell curve would be still centred around the mean but higher and more bunched in the middle so the probability in the tail greater than 83 will be less.

(d) $P(\bar{T} > 82) = 0.1$ (by symmetry)
$P(z > k) = 0.1$
Using $z \sim N(0, 1^2)$ $k = 1.28155$

Solve $1.28155 = \dfrac{82 - 80}{\dfrac{20}{\sqrt{n}}}$

$$n = 164.2$$

So 165 trucks need to be weighed.

6. (MSPEC 2020:CA18a,b)

(a) $\dfrac{365}{50} = 7.3$ g/biscuit

Std Dev. of the sample mean is $\dfrac{1.5}{\sqrt{50}}$.

Using the mean of 7.5 with the Std Dev. of $\dfrac{1.5}{\sqrt{50}}$ and the Normal Distribution $P(\bar{x} < 7.3) = 0.17$.

(b) Using $\bar{x} = \mu$ for the sample.
$P(\mu - 0.2 < \mu < \mu + 0.2) = 0.98$
$P(-0.2 < z < 0.2) = 0.98$

Now $z = \dfrac{x - \mu}{\dfrac{S}{\sqrt{n}}}$

Solve $2.3263 = \dfrac{0.2}{\left(\dfrac{1.5}{\sqrt{n}}\right)}$

$$n = 304.4$$

So 305 need to be sampled.

Alternative solution is to use $E = z \cdot \dfrac{S}{\sqrt{n}}$ with 98% C.I.

$$0.2 = 2.3263\left(\frac{1.5}{\sqrt{n}}\right)$$

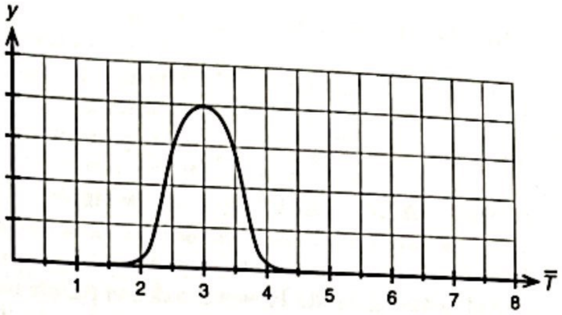
$$n \approx 305$$

7. (MSPEC 2021:CA15a,b)

(a) $\bar{T} \sim N\left(\mu = 3, \sigma_{\bar{T}} = \dfrac{2.4}{\sqrt{64}}\right)$

$P(150 < \bar{T} < 210) = 0.904$

(b)



Chapter 19: Sample Means and Confidence Intervals

1. (MSPEC 2016:CA19e)
Use a 99% confidence interval for WolliWorks to see if $\mu = 175$ L (SavaDaWater) is within C.I.

$\bar{X}$ for WolliWorks is $6570 \div 36 = 182.5$ L

$182.5 \pm \dfrac{2.576(15)}{\sqrt{36}}$

176.06 – 188.94 L
Even one of the widest (99%) C.I.'s does not contain $\mu = 175$ L (SavaDaWater).
This is significant. So, yes, claim appears to be true that WolliWorks is using more water than SavaDaWater. (Possibly wasting water)

2. (MSPEC 2017:CA13)

(a) $10{,}300 \pm 1.96 \bullet \dfrac{400}{\sqrt{100}}$

$10{,}221.6 \le \mu \le 10{,}378.4$
(b) (i) False. Not a certainty due to random sampling and variation.
(ii) False. This one cable could have been manufactured poorly by machine or operator error and not within the C.I. interval. Single observations have a larger variation than the sample mean and may fall outside the interval for the mean.
(c) Jon is incorrect. Since $n \ge 30$ the sample means for cable strengths are Normally distributed.
(d) Solve

$n = \left(\dfrac{Z \bullet \sigma}{E}\right)^2$

for Z with $n = 36$
$\sigma = 500$, E = 150
Z = 1.8
Using $Z \sim N(0, 1^2)$
$P(-1.8 < Z < 1.8) = 0.928$
$\therefore$ 92.8% C.I.

3. (MSPEC 2018:CA16)

(a) $\bar{X} = 40.62 - 0.5(1.08)$
$= 40.08$

(b) $E = z \cdot \dfrac{s}{\sqrt{n}}$

$0.54 = 2.576 \cdot \dfrac{s}{\sqrt{400}}$

$s = 4.19$

(c) (i) True. Width determined by z, s and n. Since z and n are fixed ($z = 2.576$, $n = 400$) then s is the only parameter left to affect the width. The width and s are directly proportional so an increase in s increases the width.
(ii) False. It is possible Tom's doesn't overlap due to random sampling and the variation of results this can produce. It doesn't mean that Tom's sampling 'must' be biased.
(d) With only three confidence intervals we cannot be certain which of them contain the value of μ due to the random variation of sampling results. However a clearer picture would emerge with a greater number of confidence intervals. If this was the case then the overlapping confidence intervals would be very important in determining where the population mean may lie.

4. (MSPEC 2019:CA15)
(a) 40 ± 1.645 (3)
35.06 – 44.94 minutes

(b) $\frac{S}{\sqrt{n}} = 3$

$S = 3\sqrt{n}$

November sample std deviation is $\frac{S}{\sqrt{2n}}$

$= \frac{3\sqrt{n}}{\sqrt{2n}}$

$= \frac{3}{\sqrt{2}}$

$= 2.12$ mins

(c) C. The greater the sample size ($3n$) the less the standard deviation of the Sample mean and hence the less the margin of error, meaning greater precision.
(d) We cannot be certain which one if any contain it. 10% of C.I.'s in this case will not contain the true mean.
It is more likely that A will contain it, of the three, since it is the widest (biggest standard deviation) but cannot be certain that it does. Also true mean unknown.

5. (MSPEC 2020:CA17)
(a) $\bar{x} = 175$

(b) Sample Mean Std Dev. $= \frac{S}{\sqrt{n}}$

Since $z \cdot \frac{S}{\sqrt{n}} = 25$ with $z = 1.96$

gives $\frac{S}{\sqrt{n}} = 12.76$

(c) $P(165 < \bar{x} < 185)$ is required

Std Dev. with sample size doubled $\frac{S}{\sqrt{2n}} = \frac{S}{\sqrt{2}\sqrt{n}}$

$= \frac{1}{\sqrt{2}} \cdot \frac{S}{\sqrt{n}}$

$= \frac{1}{\sqrt{2}} \cdot 12.76$

$= 9.019$

$\bar{x}$ is Normal $(175, 9.019^2)$

So $P(165 < \bar{x} < 185) = 0.7325$

6. (MSPEC 2020:CA18c)
Will use a 95% C.I. for YouBeautChokkies to see if this is higher than and does not contain the μ from BikkiesAreUs (which would be significant) to see if Charlie Chokka's claim is supported.

$\bar{x} = \frac{1090}{144} = 7.5694$ g/biscuit

Std Dev. $= \frac{1.8}{\sqrt{144}} = 0.15$

95% C.I. is 7.5694 ± 1.959(0.15)
7.275 – 7.863
μ of BikkiesAreUs of 7.5 does lie well within the interval (even 90% C.I. will likely contain it).
So Charlie Chokka is making a false claim. Appears to be spreading fake news!

7. (MSPEC 2021:CA15c)
Construct a 99% (the widest) confidence interval for μ.

Using $\bar{X} = 2.1$ and $\sigma_{\bar{X}} = \frac{2.7}{\sqrt{100}} = 0.27$, $z = 2.576$ (99%)

C.I. 2.1 ± 2.576(0.27) → (1.4, 2.8)
Sample size of 100 is large and adult $\mu = 3$ does not lie within the C.I. so the claim appears to be supported by the data.

8. (MSPEC 2021:CA17)
(a) (300, 500)
(b) $E = 100$

$100 = 1.96\left(\frac{S}{\sqrt{n}}\right)$

$\left(\frac{S}{\sqrt{n}}\right)$ = Standard deviation of sample mean $= \frac{100}{1.96} = 51.02$

(c) 200 → 50 is $\frac{1}{4}$ of the width of C.I.

$\frac{1}{4}E = \frac{1}{4}z\frac{S}{\sqrt{n}}$

$= z \cdot \frac{S}{\sqrt{16n}}$

Sample size of $16n$ required.

(d) Now $\frac{S}{\sqrt{n}} = 51.02$ from (b)

$\frac{S}{\sqrt{2n}} = \frac{1}{\sqrt{2}} \cdot \frac{S}{\sqrt{n}} = \frac{1}{\sqrt{2}}(51.02) = 36.08$

Using a Normal mean of 400 and a standard deviation of 36.08 to find $P(\bar{X} < \mu - 50 \text{ or } \bar{X} > \mu + 50)$

$= 1 - P(\mu - 50 < \bar{X} < \mu + 50) = 0.1658$

(e) There is a slim chance that none of the four may contain the true population mean. The least wide of the intervals are C and D with smaller standard deviation for D and larger sample size of $2n$ in C which cause the interval width to be less. B will be the widest as the greater the level of confidence the wider the interval. So B is the one that would appear more likely to contain the true population mean. However no guarantee it lies within any of them.
(f) (i) A → since width is affected by level of confidence. Smaller the level of confidence, 95%, the less the width.

(ii) C has width $\frac{2 \cdot zs}{\sqrt{2n}} = \frac{\sqrt{2} \cdot zs}{\sqrt{n}}$.

D has width $\frac{2z(0.85)}{\sqrt{n}} = \frac{1.6zs}{\sqrt{n}}$.

Since $\sqrt{2} \approx 1.4 < 1.6$, the smaller width interval is C.